# ARITHMETIC OF K3 SURFACES

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**Definition 0.1.** A K3 surface is a smooth projective geometrically integral surface such that

(1) $K_X \sim 0$ (canonical divisor)
(2) $H^1(X, \mathcal{O}_X) = 0$.

For $m > 0$, one has $\phi_m \colon X \dashrightarrow \mathbb{P}(H^0(X, \omega_X^{\otimes m})^\vee)$. The Kodaira dimension of $X$ is $-\infty$ if none of the $\phi_m$ exist, and $\max_{m>0} \dim \phi_m(X)$ otherwise. For a K3 surface, one has $\kappa(X) = 0$.

Consequence: K3 surfaces are not parametrizable. Proof: If there were a birational map $X \sim \mathbb{P}^2$, then $\kappa(X)$ would be negative.

Fact: If $X$ is a smooth projective geometrically irreducible surface with canonical sheaves isomorphic to $\omega_X = \mathcal{O}_X$, then either $X$ is a K3 surface or $X$ is an abelian surface.

Examples:

- Let $X$ be a smooth complete intersection $X$ of $r$ hyperplanes of degree $d_1, \dots, d_r$ in $\mathbb{P}^n$. Then $\omega_X \simeq \mathcal{O}_X(-n-1+\sum d_i)$. If we have a K3 surface, we must have $\sum d_i = n+1$ and $r = n-2$ (so that it is a surface). We may assume that $d_i \geq 2$. Then $\sum d_i \geq (n-2)2$ so $n \leq 5$. The only possibilities are
  - $n = 3$, $\vec{d} = (4)$
  - $n = 4$, $\vec{d} = (2, 3)$
  - $n = 5$, $\vec{d} = (2, 2, 2)$.

  Complete intersections have trivial $H^i(X, \mathcal{O}_X)$ for $0 < i < \dim X$, so these examples are actual K3 surfaces.
- Let $A$ be an abelian surface. Then $Y := A/[-1]$ is a singular surface with 16 ordinary double points. The blowup of $Y$ at all 16 singular points is a K3 surface containing 16 exceptional divisors each isomorphic to $\mathbb{P}^1$.
- An elliptic surface
$$y^2 + A_1(t)xy + A_3(t)y = x^3 + A_2(t)x^2 + A_4(t)x + A_6(t)$$
  in minimal Weierstrass form where $A_i \in k[t]$ satisfies $\deg A_i \leq 2i$ for all $i$ and $\deg A_i > i$ for some $i$ is a K3 surface.
- A double cover of $\mathbb{P}^2$ ramified over a sextic (with only double and triple points)
- Many other examples!

Open problem: Is there a K3 surface $X$ over a number field $k$ such that $X(k) \neq \emptyset$ and $X(k)$ is not Zariski dense? (Over finite fields $k$ the answer is yes! But are there other fields for which the answer is yes?)

The same problem is open for rational surfaces.

**Example 0.2.** The K3 surface $x^4 + y^4 + z^4 + t^4 = 0$ in $\mathbb{P}^3_{\mathbb{Q}}$ has no rational points.

**Example 0.3.** Some elliptic K3 surfaces have a Zariski dense set of rational points.

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Suppose that $X$ is a smooth projective variety over $k = \mathbb{C}$. We have

$$0 \to \mathbb{Z} \to \mathbb{C} \overset{\exp(2\pi i \cdot)}{\to} \mathbb{C}^\times \to 1$$

and hence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 1$$

in the analytic topology. Taking sheaf cohomology yields

$$\begin{array}{ccccccc}
H^1(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \mathcal{O}_X^*) & \longrightarrow & H^2(X, \mathbb{Z}) & \longrightarrow & H^2(X, \mathcal{O}_X) \\
\downarrow & & \| & & \uparrow & & \\
\operatorname{Pic}^0 X & \longrightarrow & \operatorname{Pic} X & \longrightarrow & \operatorname{NS}(X) & &
\end{array}$$

where $\operatorname{Pic}^0 X := \{[D] \in \operatorname{Pic} X : D \sim_{\text{alg}} 0\}$.

Let $X$ be a K3 surface over $\mathbb{C}$. Then

- $\operatorname{Pic}^0 X = 0$, so $\operatorname{Pic} X \simeq \operatorname{NS} X$.
- $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{22}$.

Conclusion: $\operatorname{NS} X \simeq \operatorname{Pic} X$ is a free abelian rank of rank $\leq 22$.

Fact:

- The cup product on $H^2(X, \mathbb{Z})$ coincides with the intersection pairing on $\operatorname{Pic} X$.
- Poincaré duality: $H^2(X, \mathbb{Z})$ is unimodular, with $\det = \pm 1$.

Hodge theory gives

$$H^i(X, \mathbb{C}) \simeq \bigoplus_{\substack{p,q \geq 0 \\ p+q=i}} H^q(X, \bigwedge^p \Omega_{X/\mathbb{C}}).$$

Let $H^{pq} := H^q(X, \bigwedge^p \Omega_{X/\mathbb{C}})$ and $h^{pq} := \dim H^{pq}$. We know that $H^0(X, \omega_X) = H^0(X, \mathcal{O}_X) = \mathbb{C}$ and $H^1(X, \mathcal{O}_X) = 0$. Using Serre duality and Poincaré duality, we get all the entries in the table of $h^{pq}$'s below except the middle entry (we put $h^{pq}$ in position $(p, q)$, where $0 \leq p \leq 2$ and $0 \leq q \leq 2$):

| | | |
|---|---|---|
| 1 | 0 | 1 |
| 0 | 20 | 0 |
| 1 | 0 | 1 |

The middle entry can be gotten from the fact that the topological Euler characteristic $\chi(X)$ equals $12\chi(\mathcal{O}_X) = 12 \cdot 2 = 24$. Since $\operatorname{NS} X \hookrightarrow H^{1,1}$, we have $\operatorname{rank} \operatorname{NS} \leq 20$.

Fact: The lattice $H^2(X, \mathbb{Z})$ is

- unimodular
- even (for all $x$, we have $x.x \equiv 0 \pmod 2$)
- signature $(3, 19)$, where 3 is the number of positive eigenvalues, and 19 is the number of negative eigenvalues.

These imply that $H^2(X, \mathbb{Z}) \simeq E_8(-1)^2 \oplus U^3$ as lattices, where $E_8(-1)$ is the $E_8$-lattice with inner product multiplied by $-1$, and $U$ is the rank-2 hyperbolic lattice.

On the other hand, the lattice $\operatorname{NS} X$ is

- not necessarily unimodular
- even (the adjunction formula says that for a curve $C$ on $X$, we have $2g(C) - 2 = C.(C + K) = C^2$, since $K = 0$ for a K3 surface).

- signature $(1, \rho - 1)$, where $\rho := \operatorname{rank} \mathrm{NS}$.
- primitive sublattice of $H^2(X, \mathbb{Z})$ (one says that $L \subset \Lambda$ is *primitive* if $\Lambda/L$ is torsion-free)

**Theorem 0.4** (consequence of Torelli theorem)**.** *Let $L$ be any primitive sublattice of $E_8(-1)^2 \oplus U^3$ with signature $(1, \rho - 1)$. Then there is a moduli space of K3 surfaces $X$ together with a diagram*

$$\begin{array}{ccc} L & \longrightarrow & \mathrm{NS}\, X \\ \downarrow & & \downarrow \\ E_8(-1)^2 \oplus U^3 & \longrightarrow & H^2(X, \mathbb{Z}) \end{array}$$

*compatible with $H^2(X, \mathbb{Z})$ whose irreducible components have dimension $20 - \rho$.*

**Example 0.5.** Quartic surfaces in $\mathbb{P}^3$: There are $35 = \binom{4+3}{3}$ monomials of degree 4 in 4 variables, so a quartic surface is given by a point in $\mathbb{P}^{34}$. But there is an action of $\mathrm{PGL}_4$ on this, and $\dim \mathrm{PGL}_4 = 16 - 1 = 15$. We find that $\dim \mathbb{P}^{34} / \mathrm{PGL}_4 = 19$.

**Example 0.6.** For double covers of $\mathbb{P}^2$ ramified over a sextic, one finds

$$\dim \left( \mathbb{P}^{28-1} / \mathrm{PGL}_3 \right) = 27 - (3^2 - 1) = 19.$$

**Question 0.7.** Is $\operatorname{Pic} X$ computable?

**Question 0.8.** Is there any K3 surface $X$ over any number field $k$ with $\operatorname{rank} \operatorname{Pic} \overline{X} = 1$ such that $X(k)$ is dense?

**Question 0.9.** Is there any K3 surface $X$ over any number field $k$ with $\operatorname{rank} \operatorname{Pic} \overline{X} = 1$ such that for all finite extensions $\ell$ of $k$, the set $X(\ell)$ is not dense?