COMPUTING THE PICARD RANK

RONALD VAN LUIJK

Let X_{40} be

$$y^2 = x^3 + A_2(t)x^2 + A_4(t)x + A_6(t),$$

where

$$A_2(t) = t^4 + 24t^3 + 98t^2 + 16t + 1$$
$$A_4(t) = 128t^5 + 2352t^4 + 1088t^3 + 64t^2$$
$$A_6(t) = -(512t^6 + 9216t^5 + 704t^4).$$

This parametrizes abelian surfaces with real multiplication by the quadratic order of discriminant 40.

Define α and β by $\alpha^2 + 18\alpha + 1 = 0$ and $2\beta^2 + 32\beta + 3 = 0$. The bad fibers are as follows:

Point of $\mathbb{P}^1$	Reduction type
∞	I_6
0	I_4
-1	II
$1/3$	I_2
$\alpha, \bar{\alpha}$	I_2
$\beta, \bar{\beta}$	I_3

Then

$$\sum(m_v - 1) = 5 + 3 + 0 + 1 + 2 \cdot 1 + 2 \cdot 2 = 15$$

and

$$\operatorname{rk}\operatorname{Pic} X_{\overline{\mathbb{Q}}} = \operatorname{rk} E(\overline{k}(t)) + 2 + \sum_v (m_v - 1) = \operatorname{rk} E(\overline{k}(t)) + 17 \geq 19$$

since we have points of the form $P = (x, y)$ with $x, y \in \mathbb{Q}(t)$, and $Q = (x, y\sqrt{10})$ with $x, y \in \mathbb{Q}(t)$.

To get an upper bound on $\operatorname{rk}\operatorname{Pic} X_{\overline{\mathbb{Q}}}$, use

$$\operatorname{Pic} X_{\overline{\mathbb{Q}}} \hookrightarrow \operatorname{Pic} X_{\overline{\mathbb{F}}_p} \hookrightarrow H^2_{\text{et}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell(1)).$$

All eigenvalues of Frobenius on $\operatorname{Pic} X_{\overline{\mathbb{F}}_p}$ are roots of unity. Therefore $\operatorname{rk}\operatorname{Pic} X_{\overline{\mathbb{Q}}}$ is at most the number of eigenvalues of Frobenius on the 22-dimensional space $H^2_{\text{et}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell(1))$ that are roots of unity. Each upper bound that we got this way turned out to be 20.

Let f be the characteristic polynomial of Frobenius on $H^2_{\text{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$.

Facts:

(1) $f \in \mathbb{Z}[1/p][x]$
(2) $\deg f = 22$
(3) $t^{22} f(1/t) = \pm f(t)$

Date: July 27, 2007.

(4) If $f(\alpha) = 0$, then $|\alpha| = 1$.

Non-real roots come in conjugate pairs. Real roots are ± 1. Therefore the number of root-of-unity eigenvalues is always even.

Lefschetz formula:

$$\#X(\mathbb{F}_q) = \sum_{i=0}^{4} (-1)^i \operatorname{Tr}(\operatorname{Frob} | H^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell))$$
$$= 1 - 0 + \operatorname{Tr}(\operatorname{Frob} | H^2) - 0 + q^2.$$

We will use $p = 7$ and $p = 11$. There is a known 19-dimensional $V \subset \operatorname{Pic} X_{\overline{\mathbb{Q}}} \hookrightarrow H^2(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$, so we need only the characteristic polynomial on the 3-dimensional space $W := H^2 / \operatorname{im} V$. Using symmetry, we need only one trace.

We do this for $p = 7$ and $p = 11$. For $p = 7$, the values α and β are not $\mathbb{F}_p$-rational, and we get

Point of $\mathbb{P}^1$	Number of points on fiber
∞	$6p$
0	$4p$
-1	$p+1$
$1/3$	$2(p+1)$
$\alpha, \bar{\alpha}$	not rational
$\beta, \bar{\beta}$	not rational

The nonsingular fibers have 34 points. This gives a total of 128 points over $\mathbb{F}_7$.

Thus $\operatorname{Tr}(F|H^2) = 128 - 7^2 - 1 = 78$ and $\operatorname{Tr}(F|\operatorname{im}(V \otimes \mathbb{Q}_\ell)) = 7(5+3+0+1+2+1+(-1)) = 77$. Hence $\operatorname{Tr}(F|W) = 1$. So the characteristic polynomial of F on W is $x^3 - x^2 \pm 7(-1)x \pm 7^3$: computing trace over $\mathbb{F}_{7^2}$ shows that it is

$$x^3 - x^2 - 7x + 7^3 = (x+7)(x^2 - 8x + 7^2).$$

We find that $\operatorname{Pic} X_{\overline{\mathbb{F}}_7}$ and $\operatorname{Pic} X_{\overline{\mathbb{F}}_{11}}$ have rank 20, and $\operatorname{Pic} X_{\overline{\mathbb{Q}}}$ injects into both and has rank at least 19. If the rank were not 19, then it would be of finite index in both $\operatorname{Pic} X_{\overline{\mathbb{F}}_7}$ and $\operatorname{Pic} X_{\overline{\mathbb{F}}_{11}}$. In this case, the discriminants of $\operatorname{Pic} X_{\overline{\mathbb{F}}_7}$ and $\operatorname{Pic} X_{\overline{\mathbb{F}}_{11}}$ would differ by a square factor. But they don't: the discriminants up to squares are -6 and -33, as follows from the Artin-Tate conjecture, which is proved for elliptic K3 surfaces.