# EFFECTIVE COMPUTATION OF BRAUER-MANIN OBSTRUCTIONS

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Let $k$ be a number field. Let $X$ be a smooth projective geometrically irreducible variety over $k$. Then $\operatorname{Br} X \supset \operatorname{Br}_1 X := \ker\left(\operatorname{Br} X \to \operatorname{Br} \overline{X}\right)$. We have

$$X(k) \subset X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)^{\mathrm{Br}_1} \subset X(\mathbf{A}_k).$$

There is a Brauer-Manin obstruction to the Hasse principle when $X(\mathbf{A}_k) \neq \emptyset$ but $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$.

**Question 0.1.** Can we effectively test whether $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, or whether $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$?

Cases

- $X$ is a rational surface (i.e., $\overline{X}$ is birational to $\mathbb{P}^2_{\overline{k}}$); then $\operatorname{Br}_1 X = \operatorname{Br} X$, so $X(\mathbf{A}_k)^{\mathrm{Br}_1} = X(\mathbf{A}_k)^{\mathrm{Br}}$. Also, $(\operatorname{Br} X)/(\operatorname{Br} k) \simeq H^1(G, \operatorname{Pic} \overline{X})$, which is a finite group (studied by many people, including Iskovskikh, Colliot-Thélène-Kanevsky-Sansuc, ....
- $X$ is a K3 surface: $(\operatorname{Br} X)/(\operatorname{Br} k)$ is finite (Skorobogatov-Zarhin, 2006 preprint).

There are varieties for which $(\operatorname{Br} X)/(\operatorname{Br} k)$ is infinite. It is an open question whether $(\operatorname{Br} \overline{X})^G$ is always finite.

**Theorem 0.2** (Kresch-Tschinkel)**.** *Assume that $X$ is given by equations, and that $\operatorname{Pic} \overline{X}$ is finitely generated and torsion-free with given generators. Assume that $X(k_v) \neq \emptyset$ for all $v$. Assume that we know $K$ such that the generators of $\operatorname{Pic} \overline{X}$ are defined over $K$. Then there is an effective procedure to compute $X(\mathbf{A}_K)^{\mathrm{Br}_1}$.*

*Outline of proof.*
(1) Make $\operatorname{Br}_1(X)/\operatorname{Br}(k) \simeq H^1(G, \operatorname{Pic} \overline{X})$ explicit: Given an element of $H^1(G, \operatorname{Pic} \overline{X})$, find a 2-cocycle with values in $\mathcal{O}_{X_K}(U)^\times$ representing a corresponding element of $\operatorname{Br}_1(X)$, where $U$ is some dense open subset of $X$.
(2) Repeat to obtain enough $U$'s to cover $X$.
(3) Evaluate on $k_v$-points.

□

**Example 0.3** (Cassels-Guy 1966)**.** Let $X$ be $5x^3+9y^3+10z^3+12t^3=0$. Let $k=\mathbb{Q}(\zeta)$ where $\zeta := e^{2\pi i/3}$. We have $X(k) \neq \emptyset$ if and only if $X(\mathbb{Q}) \neq \emptyset$. Let $K = k(\sqrt[3]{9/5}, \sqrt[3]{10/5}, \sqrt[3]{12/5})$. So $G := \operatorname{Gal}(K/k)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$. We find $H^1(G, \operatorname{Pic} \overline{X}) = \mathbb{Z}/3\mathbb{Z}$, and we get huge cocycles. Colliot-Thélène-Kanevsky-Sansuc manage to work within bicubic extensions, and get smaller cocycles. In fact, cubic extensions suffice: $K = k\left(\sqrt[3]{\frac{5\cdot 12}{9\cdot 10}}\right)$. One finds that $H^1(\mathbb{Z}/3\mathbb{Z}, \operatorname{Pic} X_K) = \mathbb{Z}/3\mathbb{Z}$, where $\operatorname{Pic} X_K$ has rank 3. Because the Galois group is cyclic, say generated by $\tau$, one may use the complex

$$\operatorname{Pic}(X_K) \overset{1-\tau}{\to} \operatorname{Pic}(X_K) \overset{1+\tau+\tau^2}{\to} \operatorname{Pic}(X_K) \overset{1-\tau}{\to} \cdots$$

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to compute the cohomology. Given the class of a linear combination $D'$ of exceptional curves defined over $K'$ of degree 3 over $K$, with $\mathrm{Gal}(K'/K) \simeq \langle\rho\rangle$, we want $\mathcal{O}_{X_{K'}}(D') \simeq \mathcal{L}$, where $\mathcal{L}$ is a line bundle on $X_K$. Descent:

$$\mathcal{O}_{X_{K'}(D')} \xrightarrow{\phi} \mathcal{O}_{X_{K'}({}^{\rho}D')} \xrightarrow{{}^{\rho}\phi} \mathcal{O}_{X_{K'}({}^{\rho^2}D')} \xrightarrow{{}^{\rho^2}\phi} \mathcal{O}_{X_{K'}(D')}$$

Take

$$\phi := -\frac{1}{2}(1+\sqrt[3]{15})\frac{z+\sqrt[3]{6/5}t}{x+\sqrt[3]{9/5}y}.$$

A rational section of $\mathcal{L}$ gives a twisted cubic $C \subset X_K$.

We now want to convert this 1-cocycle into a 2-cocycle with values in $K(X)^\times$. We have a linear equivalence

$$C + {}^{\tau}C + {}^{\tau^2}C \sim 3(\text{hyperplane})$$

given by some $r \in K(X)^\times$ defined uniquely up to a scalar multiple. We can choose $r$ to have coefficients in $k$. The constant $1+\sqrt[3]{15}$ and the scaling here both require satisfying a cocycle condition. The obstruction to the correct choice of scale is an element of $H^2(\mathrm{Gal}(K'/K), K'^\times)$ or $H^3(G, K^\times)$. The first obstruction vanishes since $\mathrm{Pic}\, X_K \xrightarrow{\sim} (\mathrm{Pic}\, X_{K'})^{\mathrm{Gal}(K'/K)}$ given the existence of local points. For the second obstruction, when $G$ is cyclic, we have $H^3(G, K^\times) = H^1(G, K^\times) = 0$ by Hilbert 90; more generally, the obstruction vanishes after possibly enlarging $K$.

Fact: $H^3(G, K^\times)$ is cyclic of order $\gcd\{[K:k]/[K_v/k_v]\}_v$. Inflation maps it to $H^3(\mathrm{Gal}(L/k), L^\times)$, and will be trivial when $[K:k] | [L_v:k_v]$ for some $v$. Let $\ell$ be a cyclotomic extension of $k$, and let $L = K\ell$.

Bright and Swinnerton-Dyer (2004) give an effective procedure to test an $(i+1)$-cocycle for triviality, and when trivial, to lift it to an $i$-cochain; this involves $S$-unit computations.

Summary: from the Leray spectral sequence we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\mathrm{Br}\, k \to \mathrm{Br}\, K) & \longrightarrow & \ker(\mathrm{Br}\, X \to \mathrm{Br}\, X_K) & \longrightarrow & H^1(G, \mathrm{Pic}\, X_K) & \longrightarrow & H^3(G, K^\times) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Br}\, k & \longrightarrow & \mathrm{Br}_1\, X & \longrightarrow & H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}\, \overline{X}) & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & H^1(\mathrm{Gal}(\overline{k}/K), \mathrm{Pic}\, \overline{X}) = 0 & &
\end{array}$$

The 0 at the bottom comes from $\mathrm{Pic}\, \overline{X}$ being torsion-free.

When $\mathrm{Pic}\, \overline{X}$ is not torsion-free, there exist nontrivial finite étale covers $\overline{Y} \to \overline{X}$; i.e., $\pi_1(\overline{X})^{\mathrm{alg}} \neq \{1\}$, and there is a possibility of nonabelian obstructions to the Hasse principle.