REPRESENTATION OF INTEGRAL QUADRATIC FORMS BY INTEGRAL QUADRATIC FORMS

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1. Examples

In the book on quadratic forms by Cassels, one finds the following:

Example 1.1. Let $m \equiv 3 \pmod 8$. The equation $m^2 = X^2 - 2Y^2 + 64Z^2$ has a primitive solution in $\mathbb{Z}_p$ for each p but no primitive solution in $\mathbb{Z}$. ("Primitive" means $\gcd(X, Y, Z) = 1$.)

Proof. We leave as an exercise that there exist local solutions. If $x, y, z \in \mathbb{Z}$ satisfy the equation and $\gcd(x, y, z) = 1$, then

$$(m - 8z)(m + 8z) = x^2 - 2y^2 \neq 0.$$

Suppose $p \mid m - 8z$. Then $p \neq 2$. If $\left(\frac{2}{p}\right) = 1$, then $p \equiv \pm 1 \pmod 8$. If $\left(\frac{2}{p}\right) = -1$ (so $p \equiv \pm 3 \pmod 8$), then $x \equiv y \equiv 0 \pmod p$, so $p \nmid z$, so $p \nmid m + 8z$; therefore $v_p(m - 8z)$ is even, so

$$m - 8z = \prod_{p \equiv \pm 1 \pmod 8} p^{n_p} \prod_{p \equiv \pm 3 \pmod 8} p^{2n_p} \equiv 1 \pmod 8,$$

which contradicts the hypothesis on m. □

What is going on?

Second proof. We have

$$(m - 8z)(m + 8z) = x^2 - 2y^2 \neq 0.$$

Let $\alpha = (m - 8z, 2) = (m + 8z, 2) \in \operatorname{Br} \mathbb{Q}$. We have $\alpha_{\mathbb{R}} = 0$. If $p \neq 2$, and p does not divide both $m - 8z$ and $m + 8z$, then $\alpha_{\mathbb{Q}_p} = (\text{unit}, \text{unit}) = 0 \in \operatorname{Br} \mathbb{Q}_p$. If $p \neq 2$, and p divides both $m - 8z$ and $m + 8z$, then $p | z$, and p does not divide both x and y, so $\left(\frac{2}{p}\right) = 1$, so $2 \in \mathbb{Q}_p^{\times 2}$, so $\alpha|_{\mathbb{Q}_p} = 0$. If $p = 2$, then $\alpha_{\mathbb{Q}_2} = (m - 8z, 2) = (m, 2) = (\pm 3, 2) \neq 0 \in \operatorname{Br} \mathbb{Q}_2$. This contradicts the exact sequence

$$\operatorname{Br} \mathbb{Q} \to \bigoplus_p \operatorname{Br} \mathbb{Q}_p \to \mathbb{Q}/\mathbb{Z}.$$

□

Theorem 1.2 (R. Schulze-Pillot and F. Xu)**.** *Suppose that $m, n, k \geq 1$. Then $m^2x^2 + n^{2k}y^2 - nz^2 = 1$ has no solution over $\mathbb{Z}$ if and only if*

- $(n, m) \neq 1$, *or*
- $(n, m) = 1$ *but*
  - *either* $n \equiv 5 \pmod 8$ *and* $2 \mid m$

Date: July 24, 2007.

– *or* $n \equiv 3 \pmod 8$ *and* $4 \mid m$.

2. Representing quadratic forms by quadratic forms

More generally, one can consider the following problem. Consider two quadratic forms over $\mathbb{Z}$, say g of rank n over $\mathbb{Q}$ and f of rank m over $\mathbb{Q}$, nondegenerate over $\mathbb{Q}$. Write $g \prec f$ if there exist linear forms ℓ_i with coefficients in $\mathbb{Z}$ such that

$$g(x_1, \ldots, x_n) = f(\ell_1(x_1, \ldots, x_n), \ldots, \ell_m(x_1, \ldots, x_n)).$$

In the case $n = 1$, we are asking the classical question of whether a nonzero integer a is representable as $f(x_1, \ldots, x_n)$.

In general, given a scheme $\mathcal{X}$ over $\mathbb{Z}$, we can ask whether $\mathcal{X}(\mathbb{Z}) \neq \emptyset$. Assume that over each $\mathbb{Z}_p$ we have $g \prec_{\mathbb{Z}_p} f$; does this imply $g \prec_{\mathbb{Z}} f$? This is a question of the type: does $\prod_p \mathcal{X}(\mathbb{Z}_p) \neq \emptyset$ imply $\mathcal{X}(\mathbb{Z}) \neq \emptyset$?

One reason to work with schemes: Let $\mathcal{X}_1 = \operatorname{Spec} \mathbb{Z}[x, y, z]/(f - a)$. Let $\mathcal{X} = \mathcal{X}_1 - \{x = y = z = 0\}$. Then $\mathcal{X}(\mathbb{Z})$ is the set of primitive integer solutions to $a = f(x, y, z)$.

Let $\mathcal{X}$ be a separated scheme of finite type over $\mathbb{Z}$. Let $X = \mathcal{X} \times_{\mathbb{Z}} \mathbb{Q}$. Then $\mathcal{X}(\mathbb{Z}) \hookrightarrow X(\mathbb{Q})$. Let $\mathcal{X}'$ be the schematic closure of X in $\mathcal{X}$. Fact: $\mathcal{X}'(\mathbb{Z}) = \mathcal{X}(\mathbb{Z})$ and $\mathcal{X}'(\mathbb{Z}_p) = \mathcal{X}(\mathbb{Z}_p)$. Concretely, this is saying, for instance, that $pf(x, y, z) = pa$ has the same integral solutions as $f(x, y, z) = a$.

Let k be a number field. Let $\mathcal{O} \subset k$ be the ring of integers. Let Ω be the set of places of k. Let $\mathcal{X}/\mathcal{O}$ be a separated flat scheme. Let $X = \mathcal{X} \times_{\mathcal{O}} k$. Define the adèles of X as

$$X(\mathbb{A}_k) = \bigcup_{\text{finite } S \subset \Omega} \left[\prod_{v \in S} X(k_v) \times \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v) \right] \subset \prod_{v \in \Omega} X(k_v).$$

This is the same as the set of k-morphisms $\operatorname{Spec} \mathbb{A}_k \to X$.

Over an arbitrary field k with $\operatorname{char} k = 0$, if X is a variety over k, then

$$\begin{aligned} k[X]^\times &= H^0(X, \mathbb{G}_m) \\ \operatorname{Pic} X &= H^1_{\text{Zar}}(X, \mathbb{G}_m) = H^1_{\text{ét}}(X, \mathbb{G}_m) \qquad \text{(Hilbert's theorem 90)} \\ \operatorname{Br} X &= H^2_{\text{ét}}(X, \mathbb{G}_m). \end{aligned}$$

If X/k is smooth and integral, there is an exact sequence

$$0 \to \operatorname{Br} X \to \operatorname{Br} k(X) \to \bigoplus_{\substack{Y \subset X \\ \text{irreducible codim 1}}} H^1(k(Y), \mathbb{Q}/\mathbb{Z}).$$

For X/k and $F \supset k$, we have

$$X(F) \times \operatorname{Br} X \to \operatorname{Br} F = \operatorname{Br} \operatorname{Spec} F = H^2(\mathcal{G}_F, \overline{F}^\times).$$

Now let k be a number field. Suppose $A \in \operatorname{Br} X$. We have the basic commutative diagram, where the bottom line is exact:

$$\begin{tikzcd} X(k) \arrow[r] \arrow[d, "\mathrm{ev}_A"] & X(\mathbb{A}_k) \arrow[d, "\mathrm{ev}_A"] \arrow[dr, "\theta_A"] & \\ \operatorname{Br} k \arrow[r] & \bigoplus_{v \in \Omega} \operatorname{Br} k_v \arrow[r, "\sum i_v"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

Thus (Manin 1970): $X(k) \subset X(\mathbb{A})^{\mathrm{Br}} := \bigcap_{A \in \operatorname{Br} X} \ker \theta_A$.

Analogously, we have

$$\mathcal{X}(\mathcal{O}) \subset \left(\prod \mathcal{X}(\mathcal{O}_v)\right)^{\mathrm{Br}}$$

Here Br refers to the Brauer group of X, not (only) of $\mathcal{X}$.

Let G be a connected linear algebraic group. Let X/k be a homogeneous space of G: this means that we have a group variety action $G \times X \to X$ and $G(\overline{k})$ acts transitively on $X(\overline{k})$.

Basic example: $X = G/H$ where H is a subgroup of G (note: forming the quotient variety is not a trivial operation).

Back to our general problem: Let $\mathcal{X} = \operatorname{Mor}_{\mathcal{O}}(g, f)$. Witt: Then $X = \operatorname{Mor}_k(g_k, f_k)$ is a homogeneous space of the orthogonal group $O(f_k)$.

- If $n < m$, then X is a homogeneous space of $\mathrm{SO}(f)$.
- If $n = m$ and $X(k) \neq \emptyset$, then $X = X_0 \cup X_1$ where X_0 is a homogeneous space of $\mathrm{SO}(f)$.

We are assuming $\prod \mathcal{X}(\mathcal{O}_v) \neq \emptyset$. So $\prod X(k_v) \neq \emptyset$. By Hasse's theorem (1924/25), $X(k) \neq \emptyset$. Fix a point $P_0 \in X(k)$; then $X = \mathrm{SO}(f)/H_1$, where H_1 is the stabilizer of P_0.

Suppose that $m \geq 3$. Then we can also write $\mathrm{SO}(f)/H_1 = \mathrm{Spin}(f)/H$ for some $H \leq \mathrm{Spin}(f)$. Write $f \simeq g \perp h$ over k, where f, g, h are of ranks $m, n, m-n$, respectively.

- If $m - n \geq 3$, then $H = \mathrm{Spin}(h)$.
- If $m - n = 2$, then $H = R^1_{K/k}\mathbb{G}_m$ where $K = k(\sqrt{-\det f \cdot \det g})$.
- If $m - n \leq 1$, then $H = \mu_2$, and $X = \mathrm{SO}(f)$.

General situation: Let $X = G/H$ where G is a semisimple simply connected group that is absolutely simple.

For $X = G$, we have

- $k^\times = k[G]^\times$
- $\operatorname{Pic} G = 0$
- $\operatorname{Br} k \xrightarrow{\sim} \operatorname{Br} G$.

In general,

- $k^\times \xrightarrow{\sim} k[X]^\times$.
- $\hat{H}(k) \xrightarrow{\sim} \operatorname{Pic} X$, where $\hat{H} := \operatorname{Hom}_{k\text{-groups}}(H, \mathbb{G}_m)$.
- $H^1(\mathcal{G}_k, \hat{H}(\overline{k})) \simeq \ker\left(\operatorname{Br} X \to \operatorname{Br} \overline{X}\right) / \operatorname{Br} k$, where $\overline{X} := X \times_k \overline{k}$.

If H is connected, there is an isomorphism $\operatorname{Pic} H \xrightarrow{\sim} \operatorname{Br} X / \operatorname{Br} k$: these are finite groups.

We return to the situation $g \prec f$ with g, f of ranks n, m.

- If $m - n \geq 3$, then
  - $\operatorname{Pic} X = 0$
  - $\operatorname{Br} k \xrightarrow{\sim} \operatorname{Br} X$.
- If $m - n = 2$, then
  - If $-\det f \cdot \deg g$ is a square, then
    - $\operatorname{Pic} X = \mathbb{Z}$
    - $\operatorname{Br} k = \operatorname{Br} X$.
  - If not a square, then $\operatorname{Br} X / \operatorname{Br} k = \mathbb{Z}/2\mathbb{Z}$.
- If $m - n \leq 1$, then $\operatorname{Br} X / \operatorname{Br} k = k^\times / k^{\times 2}$.

3. General theorem

Theorem 3.1. *Let k be a number field. Let $X = G/H$ where G is semisimple, simply connected and absolutely simple, and H is either connected or finite abelian. Assume that v_0 is a place of k such that $G(k_{v_0})$ is not compact: one then says that $G_{k_{v_0}}$ is "isotropic". Suppose $\mathcal{X}/\mathcal{O}$ and $X := \mathcal{X} \times_{\mathcal{O}} k \simeq G/H$. Assume that*

$$\left(\prod_{v\in\Omega} \mathcal{X}(\mathcal{O}_v)\right)^{\operatorname{Br} X} \neq \emptyset.$$

Let $\mathcal{O}_{\{v_0\}}$ be the subring of elements of k that are integral away from v_0. Then $X(\mathcal{O}_{\{v_0\}}) \neq \emptyset$.

We use the Hasse principle for semisimple simply connected groups G:

Theorem 3.2 (Eichler, Kneser, Harder, Chernousov)**.** *For a semisimple simply connected group G, the diagonal map*

$$H^1(k, G) \to \prod_{v\in\Omega} H^1(k_v, G)$$

is injective.

We also use the strong approximation theorem:

Theorem 3.3 (Eichler, Kneser, Platonov)**.** *Let G/k be semisimple simply connected and absolutely simple. If $G(k_{v_0})$ is not compact, then $G(k).G(k_{v_0})$ is dense in $G(\mathbb{A}_k)$.*

We also use

Theorem 3.4 (Kottwitz)**.** *Let H be connected. Then there is an exact sequence*

$$H^1(k, H) \to \bigoplus_{v\in\Omega} H^1(k_v, H) \to \operatorname{Hom}(\operatorname{Pic} H, \mathbb{Q}/\mathbb{Z}).$$

(The last map is constructed from the following, given for k, but which applies also to k_v:

$$H^1(k, H) \times \operatorname{Pic} H \to \operatorname{Br} k$$

defined by using $\operatorname{Ext}(H, \mathbb{G}_m) \xrightarrow{\sim} \operatorname{Pic} H$: an extension

$$1 \to \mathbb{G}_m \to E \to H \to 1$$

induces $H^1(k, H) \to H^2(k, \mathbb{G}_m) = \operatorname{Br} k$.)

One can also look at $H^1_{\text{ét}}(X, H) \times \operatorname{Pic} H \to \operatorname{Br} X$.

For μ finite abelian we have an exact sequence (Poitou, Tate)

$$H^1(k, \mu) \to {\prod}' H^1(k_v, \mu) \to \operatorname{Hom}(H^1(k, \hat{\mu}), \mathbb{Q}/\mathbb{Z})$$

where $\hat{\mu} := \operatorname{Hom}(\mu, \mathbb{G}_m)$.

Proof of Theorem 3.1. Assume H connected. Recall that $X = G/H$, so $G \to X$ is a torsor under H. We have the diagram

$$\begin{tikzcd}
G(k) \arrow[r] \arrow[d] & G(\mathbb{A}_k) \arrow[d] & \\
X(k) \arrow[r] \arrow[d] & X(\mathbb{A}_k) \arrow[r, "\text{Manin}"] \arrow[d] & \mathrm{Hom}(\mathrm{Br}\,X/\mathrm{Br}\,k, \mathbb{Q}/\mathbb{Z}) \arrow[d] \\
H^1(k,H) \arrow[r] \arrow[d] & \bigoplus_{v\in\Omega} H^1(k_v,H) \arrow[r, "\text{Kottwitz}"] \arrow[d] & \mathrm{Hom}(\mathrm{Pic}\,H, \mathbb{Q}/\mathbb{Z}) \\
H^1(k,G) \arrow[r] & \prod_{v\in\Omega} H^1(k_v,G) &
\end{tikzcd}$$

and the bottom map is injective by the Hasse principle.

Easy: If $(M_v) \in X(\mathbb{A})^{\mathrm{Br}}$, then there exist $M \in X(k)$ and $(g_v) \in G(\mathbb{A}_k)$ such that $g_v M = M_v \in X(k_v)$ for each v. Use $(M_v) \in \prod \mathcal{X}(\mathcal{O}_v)$ and the fact that $G(k_0)G(k)$ is dense in $G(\mathbb{A}_k)$ (strong approximation) to find some $g_0 \in G(k)$ such that $g_0 M \in \mathcal{X}(\mathcal{O}_v)$ for any $v \neq v_0$.

One can play the same game with G/μ for μ finite abelian, using a sequence from class field theory. $\square$

Effectivity: Can we check the hypothesis

$$\left(\prod_{v\in\Omega} \mathcal{X}(\mathcal{O}_v)\right)^{\mathrm{Br}\,X} \neq \emptyset?$$

Suppose that we are in the case where H is connected. The group $\mathrm{Pic}\,H \simeq \mathrm{Br}\,X/\mathrm{Br}\,k$ is finite for purely algebraic reasons. If one chooses $S \subset \Omega$ large enough, where $\mathcal{X} \times_{\mathcal{O}} \mathcal{O}_S \simeq \underline{G}/\underline{H}$, then it is enough to decide whether the map

$$\prod_{v\in S} \mathcal{X}(\mathcal{O}_v) \to \mathrm{Hom}(\mathrm{Pic}\,H, \mathbb{Q}/\mathbb{Z})$$

has a nontrivial kernel.

Now suppose instead that we are in the case $X = G/\mu$ with μ finite abelian. Let S be big enough for μ. Then we have the exact sequence

$$H^1_{\text{ét}}(\mathcal{O}_S, \mu) \to \prod_{v\in S} H^1(k,\mu) \to \mathrm{Hom}(H^1_{\text{ét}}(\mathcal{O}_S, \hat{\mu}), \mathbb{Q}/\mathbb{Z}).$$

Here again one may restrict attention to the kernel of the map

$$\prod_{v\in S} \mathcal{X}(\mathcal{O}_v) \to \mathrm{Hom}(H^1_{\text{ét}}(\mathcal{O}_S, \hat{\mu}), \mathbb{Q}/\mathbb{Z}),$$

but the finiteness of the group $H^1_{\text{ét}}(\mathcal{O}_S, \hat{\mu})$ comes from Dirichlet's theorem and finiteness of the class number.

4. Application to quadratic forms

We want to know whether $g \prec f$ over $\mathcal{O}$, where the ranks are n and m, with $m \geq 3$.

If $m - n \geq 3$, then $\operatorname{Br} X / \operatorname{Br} k = 0$. Then $\prod_{v \in \Omega} \mathcal{X}(\mathcal{O}_v) \neq \emptyset$ implies $\mathcal{X}(\mathcal{O}_{\{v_0\}}) \neq \emptyset$ if $f_{k_{v_0}}$ is isotropic; over $\mathbb{Q}$, if we can take $v_0 = \infty$, then we get an integral representation.

Suppose $m - n = 2$. Consider $m = 3$, $n = 1$. We want to solve $a = f(x, y, z)$ with $a \neq 0$. This defines $\mathcal{X}$. Let $X = \mathcal{X} \times_{\mathcal{O}} k$. Then $\operatorname{Br} X / \operatorname{Br} k$ is 0 if $d := -a \cdot \det f$ is a square, and $\mathbb{Z}/2\mathbb{Z}$ if d is not a square. Consider the latter case.

$$\prod_{v \in \Omega} \mathcal{X}(\mathcal{O}_v) \to \mathbb{Z}/2\mathbb{Z}$$

$$(M_v) \mapsto \sum_v \operatorname{ev}_A(M_v).$$

How to find $A \in \operatorname{Br} X$? Since $\prod \mathcal{X}(\mathcal{O}_v) \neq \emptyset$, we have $\prod X(k_v) \neq \emptyset$, so we can find a point $P_0 \in X(k)$. Let $Y \subset \mathbb{P}^3_k$ be defined by $q(x, y, z) - at^2 = 0$. Let $0 = \ell_1(x, y, z, t)$ be the tangent plane to Y at P_0. We can show that $f(x, y, z) - at^2 = \ell_1 \ell_2 + c(\ell_3^2 - \ell_4^2)$. Define $A \in \operatorname{Br} k(X)$ by $A = \left(\frac{\ell_1(x,y,z,t)}{t}, d \right)$. We check that $A \in \operatorname{Br} X \setminus \operatorname{Br} k$. Let $K = k(\sqrt{d})$. Check the kernel of the map Θ obtained as the composition

$$\prod_{v \in \Omega} \mathcal{X}(\mathcal{O}_v) \to \bigoplus_{v \in \Omega} \frac{k_v^\times}{N K_v^\times} \to \mathbb{Z}/2\mathbb{Z}$$

where the first map sends M_v to $(\ell_1/t)(M_v)$. Assuming there exists an archimedean v_0 where f_{v_0} is isotropic, we have $\mathcal{X}(\mathcal{O}) \neq \emptyset$ if and only if there is a point in the kernel of Θ.

Let us apply this to

$$m^2 x^2 + n^{2k} y^2 - n z^2 = 1.$$

This is solvable over each $\mathbb{Z}_p$ if and only if $(n, m) = 1$; let us assume this. There is an obvious rational point, namely $P_0 := (0, -1/n^k, 0)$. Write the equation as

$$(1 + n^k y)(1 - n^k y) = m^2 x^2 - n z^2.$$

The tangent plane at P_0 is $1 + n^k y = 0$. In $\operatorname{Br} X$, we have $A = (1 + n^k y, n)$ (the number n is the old d). We have

$$\mathcal{X}(\mathbb{Z}_p) \to \operatorname{Br} \mathbb{Q}_p.$$

If $p \neq 2$, then $\operatorname{ev}_A(\mathcal{X}(\mathbb{Z}_p)) = 0$ always. If $p = 2$, then $\operatorname{ev}_A(\mathcal{X}(\mathbb{Z}_2)) = 1$ in $\mathbb{Z}/2\mathbb{Z}$ if and only if $n \equiv 5 \pmod 8$ and $2 \mid m$, or $n \equiv 3 \pmod 8$ and $4 \mid m$.

Exercise 4.1 (Schulze-Pillot). Take $k = \mathbb{Q}(\sqrt{35})$. If p is a prime such that $\left(\frac{p}{7}\right) = 1$, then $7p^2 = a^2 + b^2 + c^2$ over each $\mathcal{O}_v$ but not over $\mathcal{O} = \mathbb{Z}[\sqrt{35}]$. Prove that this is given by a Brauer-Manin obstruction.

Exercise 4.2. Fix $f(x, y, z)$. The elements $a \in \mathbb{Z}$ such that $a \prec f$ over each $\mathbb{Z}_p$ but not over $\mathbb{Z}$ fall into finitely many classes in $\mathbb{Q}^\times / \mathbb{Q}^\times$. (The same holds over any number field.)

We now consider the case $m = n + 2$ with $m \geq 3$. So $X = \operatorname{Spin}(f)/T$ where $T := R^1_{K/k} \mathbb{G}_m$ is given by an equation $N_{K/k}(\ ) = 1$, where $K = k(\sqrt{d})$ (which we assume is a field), where

$d := -\det f \cdot \det g$. We have $\operatorname{Pic} T = \mathbb{Z}/2\mathbb{Z}$. What, concretely, is the map

$$\begin{tikzcd}
\prod_{v\in\Omega} X(k_v) \arrow[r] \arrow[dr] & \bigoplus_{v\in\Omega} H^1(k_v, T) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(\operatorname{Pic} T, \mathbb{Q}/\mathbb{Z}) \arrow[d, equal] \\
& k_v^\times / N K_v^\times \arrow[r] & \mathbb{Z}/2\mathbb{Z}?
\end{tikzcd}$$

Use

$$\begin{tikzcd}
& & 1 \arrow[d] & & \\
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & T_1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_2 \arrow[r] & \mathrm{Spin}(f) \arrow[r] \arrow[d] & \mathrm{SO}(f) \arrow[r] \arrow[d] & 1 \\
& & X \arrow[r, equal] \arrow[d] & X & \\
& & 0 & &
\end{tikzcd}$$

where $T_1 \simeq T$. We have

$$\begin{tikzcd}
\mathrm{SO}(f)(F) \arrow[rr] \arrow[d] & & F^\times / F^{\times 2} \arrow[d] \\
X(F) \arrow[r] & H^1(F, T) \arrow[r, equal] & F^\times / N(F.K)^\times
\end{tikzcd}$$

where the top map is the spinor norm sending a product of (an even number of) reflections $\prod \tau_{v_i}$ to $\prod f(v_i)$.

Application to an example of Siegel:

$$x^2 + 32y^2 \prec x^2 + 128y^2 + 128yz + 544z^2 - 64t^2$$

over each $\mathbb{Z}_p$ but not over $\mathbb{Z}$.

Classical problem: Suppose we have a quadratic space $(V/k, f_k)$ with f_k nondegenerate, and we have $N, M \subset V$ where M is a full lattice: $N_k \subset V$ and $M_k = V$. Assume that $f(M) \subset \mathcal{O}$ and $f(N) \subset \mathcal{O}$, and that $g := f|_{N_k}$ is nondegenerate. Let $\operatorname{Hom}((N, g), (M, f))(A)$ be the set of linear $\phi\colon N_A \to M_A$ such that $\phi^*(f) = g$. Define $\mathcal{X} = \mathbf{Hom}((N, g), (M, f))$. We are given $P_0 \in X(k)$. The group $O(f)(\mathbb{A})$ acts on the full lattices in (V, f_k).

"N is represented by the proper class of M" translates as $\mathcal{X}(\mathcal{O}) \neq \emptyset$.

"N is represented by the genus of M" translates as $\prod_v \mathcal{X}(\mathcal{O}_v) \neq \emptyset$.

"N is represented by the proper spinor genus of M" translates as $\left(\prod_v \mathcal{X}(\mathcal{O}_v)\right)^{\operatorname{Br} X} \neq \emptyset$.

There is also a strong approximation statement analogous to our Brauer-Manin obstruction statement for the integral Hasse principle.