# $p$-adic Analysis in Arithmetic Geometry

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# 1. Introduction

We all know the fields $\mathbb{R}$ and $\mathbb{C}$ of real and complex numbers as the completion of the field $\mathbb{Q}$ of rational numbers and its algebraic closure. In particular, we have a canonical embedding $\mathbb{Q} \hookrightarrow \mathbb{R} \subset \mathbb{C}$, which we can sometimes use to get number theoretic results by applying analysis over $\mathbb{R}$ or $\mathbb{C}$.

Now $\mathbb{R}$ is not the only completion of $\mathbb{Q}$. Besides the usual absolute value, there are more absolute values on $\mathbb{Q}$; to be precise, up to a natural equivalence (and except for the trivial one), there is one absolute value $|\cdot|_p$ for each prime number $p$. (We will explain what an absolute value is in due course.) Completing $\mathbb{Q}$ with respect to $|\cdot|_p$ leads to the field $\mathbb{Q}_p$ of $p$-adic numbers; we can then take its algebraic closure $\bar{\mathbb{Q}}_p$ and the completion $\mathbb{C}_p$ of that ($\bar{\mathbb{Q}}_p$ is, in contrast to $\bar{\mathbb{R}} = \mathbb{C}$, not complete), which has similar properties as $\mathbb{C}$: it is the smallest extension field of $\mathbb{Q}$ that is algebraically closed and complete with respect to the $p$-adic absolute value.

There is a general philosophy in Number Theory that 'all completions are created equal' and should have the same rights. In many situations, one gets the best results by considering them all together. Since $\mathbb{Q}_p$ and $\mathbb{C}_p$ are complete metric spaces (and the former is, like $\mathbb{R}$ or $\mathbb{C}$, locally compact), we can try to do analysis over them. Many concepts and results of 'classical' analysis carry over without great problems. But we will see that there is one feature of the $p$-adic topology that is a stumbling block for an easy transfer of certain parts of analysis (like for example line integrals) to the $p$-adic setting: this topology is totally disconnected. One goal of this lecture course is to explain a way to resolve this problem, which is to embed $\mathbb{C}_p$ (say) into a larger 'analytic space' $\mathbb{C}_p^{\mathrm{an}}$ that is (in this case, even uniquely) path-connected. This approach is due to Vladimir Berkovich; the analytic spaces constructed in this way are also known as *Berkovich Spaces*.

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## 2. $p$-ADIC NUMBERS

In this section we will recall (or introduce) the field $\mathbb{Q}_p$ of $p$-adic numbers and its properties. We begin with a rather general definition.

2.1. **Definition.** Let $K$ be a field. An *absolute value* on $K$ is a map **DEF** absolute value

$$|\cdot| \colon K \longrightarrow \mathbb{R}_{\ge 0}, \quad x \longmapsto |x|$$

with the following properties.

(1) $|x| = 0 \iff x = 0$.

(2) (Multiplicativity) $|xy| = |x| \cdot |y|$.

(3) (Triangle Inequality) $|x + y| \le |x| + |y|$.

If $|\cdot|$ satisfies the stronger inequality

(3′) (Ultrametric Triangle Inequality) $|x + y| \le \max\{|x|, |y|\}$,

then the absolute value is said to be *ultrametric* or *non-archimedean*, otherwise it is *archimedean*.

Two absolute values $|\cdot|_1$ and $|\cdot|_2$ on $K$ are said to be *equivalent*, if there is a constant $c > 0$ such that $|x|_2 = |x|_1^c$ for all $x \in K$. ♢

Properties (1), (2) and (3) imply that $d(x, y) = |x - y|$ defines a *metric* on $K$. ((2) is needed for the symmetry: it implies that $|-1| = 1$, so that $|y - x| = |-1| \cdot |x - y| = |x - y|$.) The absolute value is then continuous as a real-valued function on the metric space $(K, d)$.

One can show that two absolute values on $K$ are equivalent if and only if they induce the same topology on $K$ (Exercise).

2.2. **Lemma.** *Let $(K, |\cdot|)$ be a field with a non-archimedean absolute value. Then $R = \{x \in K : |x| \le 1\}$ is a subring of $K$. $R$ is a local ring (i.e., it has exactly one maximal ideal) and has $K$ as its field of fractions.* **LEMMA** valuation ring

*Proof.* Exercise! ❑

This ring is the *valuation ring* of $(K, |\cdot|)$. **DEF** valuation ring

Recall the following definition.

2.3. **Definition.** A metric space $(X, d)$ is said to be *complete*, if every Cauchy sequence in $X$ converges in $X$: if $(x_n)$ is a sequence in $X$ such that **DEF** complete metric space

$$\lim_{n\to\infty} \sup_{m\ge 1} d(x_n, x_{n+m}) = 0\,,$$

then there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$.

If $K$ is a field with absolute value $|\cdot|$, then $(K, |\cdot|)$ is said to be *complete*, if the metric space $(K, d)$ is, where $d$ is the metric induced by the absolute value. ♢

**2.4. Examples.** **EXAMPLES** absolute values

Every field $K$ has the *trivial absolute value* $|\cdot|_0$ with $|x|_0 = 1$ for all $x \neq 0$.

The usual absolute value is an (archimedean) absolute value on $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. The latter two are complete, $\mathbb{Q}$ is not.

$|f(x)/g(x)| = e^{\deg(f)-\deg(g)}$ defines a non-archimedean absolute value on $K(x)$, the field of rational functions in one variable over $K$. (Exercise!)

If $|\cdot|$ is an absolute value and $0 < c \leq 1$, then $|\cdot|^c$ is also an absolute value, which is equivalent to $|\cdot|$. If $|\cdot|$ is non-archimedean, then this remains true for $c > 1$. (Exercise!) ♣

We can now introduce the $p$-adic absolute values on $\mathbb{Q}$.

**2.5. Definition.** Let $p$ be a prime number. If $a \neq 0$ is an integer, we define its *$p$-adic valuation* to be **DEF** $p$-adic absolute value

$$v_p(a) = \max\{n \in \mathbb{Z}_{\geq 0} : p^n \mid a\}\,.$$

For $a = r/s \in \mathbb{Q}^\times$ (with $r, s \in \mathbb{Z}$, $s \neq 0$), we set $v_p(a) = v_p(r) - v_p(s)$. The *$p$-adic absolute value* on $\mathbb{Q}$ is given by

$$|x|_p = \begin{cases} 0 & \text{if } x = 0, \\ p^{-v_p(x)} & \text{otherwise.} \end{cases}$$ ♢

So the $p$-adic absolute value of $x \in \mathbb{Q}$ is small when (the numerator of) $x$ is divisible by a high power of $p$, and it is large when the denominator of $x$ is divisible by a high power of $p$. If $x \in \mathbb{Z}$, then we clearly have $|x|_p \leq 1$: contrary to the familiar situation with the usual absolute value, the integers form a bounded subset of $\mathbb{Q}$ with respect to $|\cdot|_p$! This explains the word 'non-archimedean' — the *Archimedean Axiom* states that if $x, y \in \mathbb{R}_{>0}$, then there is $n \in \mathbb{Z}$ such that $nx > y$; this is equivalent to saying that $\mathbb{Z}$ is unbounded.

**2.6. Lemma.** *The $p$-adic absolute value is a non-archimedean absolute value on $\mathbb{Q}$.* **LEMMA** $|\cdot|_p$ is abs. value

*Proof.* We check the properties in Definition 2.1. Property (1) is clear from the definition. Property (2) follows from $v_p(ab) = v_p(a) + v_p(b)$, which is a consequence of unique factorization. Property (3′) follows from $v_p(a + b) \geq \min\{v_p(a), v_p(b)\}$, which is a consequence of the elementary fact that $p^n \mid a$ and $p^n \mid b$ together imply that $p^n \mid a + b$. ❑

Note that $\mathbb{Q}$ is not complete with respect to $|\cdot|_p$ (Exercise!).

**2.7. Example.** Let $(K, |\cdot|)$ be a field with absolute value. Then for every $x \in K$ with $|x| < 1$, the series **EXAMPLE** geometric series

$$\sum_{n=0}^{\infty} x^n$$

converges to the limit $1/(1 - x)$: we have

$$\Big|\sum_{n=0}^{N-1} x^n - \frac{1}{1-x}\Big| = \Big|\frac{-x^N}{1-x}\Big| = |1 - x|^{-1} \cdot |x|^N\,,$$

which tends to zero, since $|x| < 1$ (note that $|1 - x| \geq 1 - |x| > 0$, so the fraction makes sense).

For example,

$$1 + p + p^2 + p^3 + p^4 + \ldots = \frac{1}{1 - p}$$

in $(\mathbb{Q}, |\cdot|_p)$. ♣

From the viewpoint of analysis, it is desirable to work with a complete field. Indeed, one possible construction of the field of real numbers is as the smallest complete field containing $(\mathbb{Q}, |\cdot|_\infty)$, where (from now on) $|\cdot|_\infty$ denotes the usual absolute value. In fact, this construction works quite generally.

2.8. **Theorem.** *Let $(K, |\cdot|)$ be a field with an absolute value. Then there is a field $(K', |\cdot|')$ extending $K$ such that $|\cdot|'$ restricts to $|\cdot|$ on $K$, $(K', |\cdot|')$ is complete and $K$ is dense in $K'$.* **THM** completion

*Proof.* Let $C(K)$ be the ring of Cauchy sequences over $K$ (with term-wise addition and multiplication). The set $N(K)$ of null sequences (i.e., sequences converging to zero) forms an ideal in the ring $C(K)$. We show that this ideal is actually maximal. It does not contain the unit (all-ones) sequence, so it is not all of $C(K)$. If $(x_n) \in C(K) \setminus N(K)$, then it follows from the definition of 'Cauchy sequence' that there are $n_0 \in \mathbb{Z}_{>0}$ and $c > 0$ such that $|x_n| \geq c$ for all $n \geq n_0$. Then the sequence $(y_n)$ given by $y_n = 0$ for $n < n_0$ and $y_n = 1/x_n$ for $n \geq n_0$ is a Cauchy sequence and $(1) = (x_n) \cdot (y_n) + (z_n)$ with $(z_n) \in N(K)$, so $N(K) + C(K) \cdot (x_n) = C(K)$.

Since $N(K)$ is a maximal ideal in $C(K)$, the quotient ring $K' := C(K)/N(K)$ is a field. There is a natural inclusion $K \hookrightarrow C(K)$ by mapping $a \in K$ to the constant sequence $(a)$, which by composition with the canonical epimorphism $C(K) \to K'$ gives an embedding $i \colon K \hookrightarrow K'$. We define $|\cdot|'$ by

$$|[(x_n)]|' = \lim_{n \to \infty} |x_n|$$

(where $[(x_n)]$ denotes the residue class mod $N(K)$ of the sequence $(x_n) \in C(K)$). The properties of absolute values and of Cauchy sequences imply that this is well-defined (i.e., the limit exists and does not depend on the choice of the representative sequence). That $|\cdot|'$ is an absolute value follows easily from the assumption that $|\cdot|$ is, and it is clear that $|\cdot|'$ restricts to $|\cdot|$ on $K$.

It is also easy to see that $K$ is dense in $K'$. Let $x = [(x_n)] \in K'$, then $|i(x_n) - x|' = \lim_{m \to \infty} |x_n - x_{n+m}|$ tends to zero as $n$ tends to infinity, so we can approximate $x$ arbitrarily closely by elements of $K$.

It remains to show that $K'$ is complete. So let $(x^{(\nu)})_\nu$ be a Cauchy sequence in $K'$ and represent each $x^{(\nu)}$ by a Cauchy sequence $(x_n^{(\nu)})_n$ in $K$. Since $K$ is dense in $K'$, we can do this in such a way that $|i(x_n^{(\nu)}) - x^{(\nu)}|' \leq 2^{-n}$ for all $\nu$ and $n$. (The reason for this requirement is that we need some uniformity of convergence to make the proof work.) Let $y_n = x_n^{(n)}$. We claim that $(y_n)$ is a Cauchy sequence (in $K$) and that $y = [(y_n)] = \lim_{\nu \to \infty} x^{(\nu)}$ (in $K'$). To see the first, pick $\varepsilon > 0$. There is $\nu_0$ such that $|x^{(\nu)} - x^{(\nu+\mu)}|' < \varepsilon$ for all $\nu \geq \nu_0$ and $\mu \geq 0$. Then

$$\begin{aligned}
|y_n - y_{n+m}| &= |x_n^{(n)} - x_{n+m}^{(n+m)}| \\
&\leq |i(x_n^{(n)}) - x^{(n)}|' + |x^{(n)} - x^{(n+m)}|' + |x^{(n+m)} - i(x_{n+m}^{(n+m)})|' \\
&\leq 2^{-n} + \varepsilon + 2^{-n-m} < 2\varepsilon
\end{aligned}$$

when $2^{-n} < \varepsilon/2$ and $n \geq \nu_0$. To see the second, note that

$$\begin{aligned}|x^{(\nu)} - y|' &= \lim_{n\to\infty} |x_n^{(\nu)} - x_n^{(n)}| \\ &\leq \limsup_{n\to\infty} \bigl(|i(x_n^{(\nu)}) - x^{(\nu)}|' + |x^{(\nu)} - x^{(n)}|' + |x^{(n)} - i(x_n^{(n)})|'\bigr) \\ &\leq \limsup_{n\to\infty} (2^{-n} + |x^{(\nu)} - x^{(n)}|' + 2^{-n}) = \limsup_{n\to\infty} |x^{(\nu)} - x^{(n)}|' \,,\end{aligned}$$

which tends to zero as $\nu \to \infty$. ❑

One can show that $(K', |\cdot|')$ is determined up to unique isomorphism (of extensions of $K$ with absolute value), but we will not need this in the following.

Since $|\cdot|'$ extends $|\cdot|$, we will usually use the same notation for both. We will also consider $K$ as a subfield of $K'$.

2.9. **Definition.** Let $p$ be a prime number. The completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the $p$-adic absolute value is called the *field of $p$-adic numbers*. Its valuation ring $\mathbb{Z}_p$ is the *ring of $p$-adic integers*. ♢

**DEF**
field of $p$-adic numbers

Since $|x|_p$ takes a discrete set of values for $x \neq 0$, it follows that

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} = \{x \in \mathbb{Q}_p : |x|_p < p\}$$

is closed *and* open in $\mathbb{Q}_p$. This implies that for any two distinct elements $x, y \in \mathbb{Q}_p$, there are disjoint open neighborhoods $X$ of $x$ and $Y$ of $y$ such that $\mathbb{Q}_p = X \cup Y$: $\mathbb{Q}_p$ is *totally disconnected*. To see this, let $\delta = |x - y|_p > 0$. Then the open ball $X$ around $x$ of radius $\delta$ (which is $x + p^{n+1}\mathbb{Z}_p$, if $\delta = p^{-n}$) is also closed, so its complement $Y = \mathbb{Q}_p \setminus X$ is open as well, and $y \in Y$. One consequence of this is that any continuous map $\gamma\colon [0,1] \to \mathbb{Q}_p$ is constant (otherwise let $x$ and $y$ be two distinct elements in the image and let $X$ and $Y$ be as above; then $[0,1]$ is the disjoint union of the two open non-empty subsets $\gamma^{-1}(X)$ and $\gamma^{-1}(Y)$, contradicting the fact that intervals are connected).

Next we want to show that $\mathbb{Z}_p$ is compact. For this we need a fact about approximation of $p$-adic numbers by rationals.

2.10. **Lemma.** *Let $x \in \mathbb{Q}_p \setminus \{0\}$ with $|x|_p = p^{-n}$. There is $a \in \{0, 1, 2, \ldots, p-1\}$ such that $|x - ap^n|_p < |x|_p$.*

**LEMMA**
approximation of $p$-adic numbers

*Proof.* Replacing $x$ by $p^{-n}x$, we can assume that $|x|_p = 1$, in particular, $x \in \mathbb{Z}_p$. Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, there is $r/s \in \mathbb{Q}$ such that $|x - r/s|_p < 1$. By the ultrametric triangle inequality, $|r/s|_p \leq 1$, so that we can assume that $p \nmid s$. Let $a \in \{0, 1, \ldots, p-1\}$ be such that $as \equiv r \bmod p$. Then

$$|x - a|_p = \bigl|(x - r/s) + (r/s - a)\bigr|_p \leq \max\{|x - r/s|_p, |r/s - a|_p\} < 1 \,,$$

since $|r/s - a|_p = |r - as|_p/|s|_p < 1$. ❑

It follows that that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$: iterating the statement of the lemma, we find that for $x \in \mathbb{Z}_p$ and $n > 0$ there is $a \in \mathbb{Z}$ (with $0 \leq a < p^n$) such that $|x - a|_p < p^{-n}$.

2.11. **Theorem.** *The ring $\mathbb{Z}_p$ is compact with respect to the metric induced by the $p$-adic absolute value. In particular, $\mathbb{Q}_p$ is locally compact.* **THM** $\mathbb{Z}_p$ is compact

*Proof.* We show that any sequence $(x_n)$ in $\mathbb{Z}_p$ has a convergent subsequence. Since $\mathbb{Z}_p$ is complete ($\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \le 1\}$ is a closed subset of the complete space $\mathbb{Q}_p$), it is enough to show that there is a subsequence that is a Cauchy sequence. We do this iteratively. Let $n_1$ be the smallest index $n$ such that there are infinitely many $m > n$ with $|x_n - x_m|_p < 1$. Such an $n$ must exist by Lemma 2.10, which implies that there is some $a$ such that $|x_n - a|_p < 1$ for infinitely many $n$. Now let $n_2$ be the smallest index $n > n_1$ such that $|x_n - x_{n_1}|_p < 1$ and such that there are infinitely many $m > n$ with $|x_n - x_m| < p^{-1}$. This again exists by Lemma 2.10, where we restrict to the infinitely many $n$ such that $|x_n - x_{n_1}|_p < 1$. We continue in this way: $n_{k+1}$ is the smallest $n > n_k$ such that $|x_n - x_{n_k}|_p < p^{-(k-1)}$ and such that there are infinitely many $m > n$ with $|x_n - x_m|_p < p^{-k}$. Then $|x_{n_{k+1}} - x_{n_k}|_p < p^{-(k-1)}$ for all $k \ge 1$, which by the ultrametric triangle inequality implies that $(x_{n_k})_k$ is a Cauchy sequence:

$$\left|x_{n_{k+l}} - x_{n_k}\right|_p \le \max\left\{\left|x_{n_{k+m}} - x_{n_{k+m-1}}\right|_p : 1 \le m \le l\right\} < p^{-(k-1)}.$$

That $\mathbb{Q}_p$ is locally compact follows, since any $x \in \mathbb{Q}_p$ has the compact neighborhood $x + \mathbb{Z}_p$. ❑

2.12. **Definition.** Let $(K, |\cdot|)$ be a complete field with a non-archimedean absolute value. By Lemma 2.2 the valuation ring $R = \{x \in K : |x| \le 1\}$ is a local ring with unique maximal ideal $M = \{x \in K : |x| < 1\}$. The quotient ring $k = R/M$ is therefore a field, the *residue field* of $K$. The canonical map $R \to k$ is called the *reduction map* and usually denoted $x \mapsto \bar{x}$. ♢ **DEF** residue field

Before we continue we note an easy but important fact on non-archimedean absolute values.

2.13. **Lemma.** *Let $(K, |\cdot|)$ be a field with a non-archimedean absolute value and let $x, y \in K$ with $|x| > |y|$. Then $|x + y| = |x|$.* **LEMMA** all triangles are isosceles

*If we have $x_1, x_2, \ldots, x_n \in K$ such that $x_1 + x_2 + \ldots + x_n = 0$ and $n \ge 2$, then there are at least two indices $1 \le j < k \le n$ such that*

$$|x_j| = |x_k| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

*Proof.* We have $|x + y| \le \max\{|x|, |y|\} = |x|$. Assume that $|x + y| < |x|$. Then

$$|x| = |(x + y) + (-y)| \le \max\{|x + y|, |-y|\} < |x|,$$

a contradiction.

For the second statement observe that if we had just one $j$ such that $|x_j|$ is maximal, then by the second statement we would have

$$0 = |x_1 + x_2 + \ldots + x_n| = \left|x_j + \sum_{k \ne j} x_k\right| = |x_j|,$$

which is a contradiction, since $x_j \ne 0$ in this situation. ❑

If $|\cdot|$ is a non-archimedean absolute value on a field $K$, then we can extend it to the polynomial ring $K[x]$.

2.14. **Lemma.** *Let $(K, |\cdot|)$ be a field with a non-archimedean absolute value. For $f = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \in K[x]$ we set*

**LEMMA** extension of abs. value to $K[x]$

$$|f| := \max\{|a_0|, |a_1|, |a_2|, \ldots, |a_n|\}\,.$$

*Then properties (1), (2) and (3′) of Definition 2.1 are satisfied for elements of $K[x]$.*

*If $K$ is complete with respect to $|\cdot|$, then for each $n \in \mathbb{Z}_{\geq 0}$, the space $K[x]_{<n}$ of polynomials of degree $< n$ is a complete metric space for the metric induced by $|\cdot|$ on $K[x]$.*

*Proof.* Exercise. ❑

The following is an important tool when working in complete non-archimedean fields.

2.15. **Theorem.** *Let $(K, |\cdot|)$ be a complete non-archimedean field with valuation ring $R$ and residue field $k$. Let $F \in R[x]$ be a polynomial such that $|F| = 1$ and suppose that we have a factorization $\bar{F} = f_1f_2$ in $k[x]$ such that $f_1$ is monic and $f_1$ and $f_2$ are coprime. (The notation $\bar{F}$ means the polynomial in $k[x]$ whose coefficients are obtained by reduction from those of $F$.) Then there are unique polynomials $F_1, F_2 \in R[x]$ such that $F = F_1F_2$, $F_1$ is monic and $\bar{F}_1 = f_1$, $\bar{F}_2 = f_2$.*

**THM** Hensel's Lemma

*Proof.* Let $n = \deg F$, $n_1 = \deg f_1$, $n_2 = n - n_1 \geq \deg f_2$. We write $R[x]_{<n}$ for the $R$-module of polynomials of degree $< n$.

We choose polynomials $F_1^{(0)}, F_2^{(0)} \in R[x]$ such that $\deg F_1^{(0)} = n_1$, $\deg F_2^{(0)} = n_2$, $\bar{F}_1^{(0)} = f_1$, $\bar{F}_2^{(0)} = f_2$, $F_1^{(0)}$ is monic and the leading coefficient of $F_2^{(0)}$ is the same as that of $F$; then $F - F_1^{(0)}F_2^{(0)} \in R[x]_{<n}$ and

$$\delta := |F - F_1^{(0)}F_2^{(0)}| < 1\,.$$

We claim that the $k$-linear map

$$k[x]_{<n_1} \times k[x]_{<n_2} \longrightarrow k[x]_{<n}\,, \qquad (h_1, h_2) \longmapsto h_1f_2 + h_2f_1$$

is an isomorphism: it is injective, since $h_1f_2 + h_2f_1 = 0$ implies that $f_1$ divides $h_1f_2$, which in turn implies that $f_1$ divides $h_1$ (since $f_1$ and $f_2$ are coprime). But $\deg h_1 < n_1 = \deg f_1$, so $h_1 = 0$. Since $f_1 \neq 0$, it follows that $h_2 = 0$, too. Finally we observe that the dimensions of source and target are the same.

If $M$ is the matrix representing this linear map with respect to the $k$-bases $((1, 0), (x, 0), \ldots (x^{n_1-1}, 0), (0, 1), (0, x), \ldots, (0, x^{n_2-1}))$ and $(1, x, x^2, \ldots, x^{n_1+n_2-1})$, then $\det(M) \neq 0$. Let now

$$\Phi\colon R[x]_{<n_1} \times R[x]_{<n_2} \longrightarrow R[x]_{<n}\,, \qquad (H_1, H_2) \longmapsto H_1F_2^{(0)} + H_2F_1^{(0)}\,.$$

Its matrix $\tilde{M}$ with respect to the 'power bases' reduces to $M$, so $|\det(\tilde{M})| = 1$, which means that $\tilde{M}$ is in $\mathrm{GL}(n, R)$, and $\Phi$ is invertible. It also follows that if $(H_1, H_2) = \Phi^{-1}(H)$, then $|H_1|, |H_2| \leq |H|$.

We now want to find $F_1$ and $F_2$ by adjusting $F_1^{(0)}$ and $F_2^{(0)}$. So we would like to determine $H_1 \in R[x]_{<n_1}$ and $H_2 \in R[x]_{<n_2}$ with $|H_1|, |H_2| < 1$ such that

$$F = (F_1^{(0)} + H_1)(F_2^{(0)} + H_2) = F_1^{(0)}F_2^{(0)} + H_1F_2^{(0)} + H_2F_1^{(0)} + H_1H_2\,.$$

We ignore the nonlinear term $H_1H_2$ and choose $(H_1, H_2)$ such that the linear terms correct the mistake, i.e., $(H_1, H_2) = \Phi^{-1}(F - F_1^{(0)}F_2^{(0)})$. From the above we know that $|H_1|, |H_2| \le \delta$ and therefore

$$\big|F - (F_1^{(0)} + H_1)(F_2^{(0)} + H_2)\big| = |H_1H_2| \le \delta^2 .$$

Repeating this with the new approximations $F_j^{(1)} = F_j^{(0)} + H_j$ we obtain $F_j^{(2)}$ such that $|F_j^{(1)} - F_j^{(2)}| \le \delta^2$ and $|F - F_1^{(2)}F_2^{(2)}| \le \delta^4$. Iterating this procedure, we construct sequences $(F_j^{(m)})_{m\ge 0}$ in $R[x]_{<n_j}$ such that

$$|F_1^{(m)} - F_1^{(m+1)}| \le \delta^{2^m} , \quad |F_2^{(m)} - F_2^{(m+1)}| \le \delta^{2^m} \quad \text{and} \quad |F - F_1^{(m)}F_2^{(m)}| \le \delta^{2^m}$$

for all $m$. Since $R[x]_{<n_j}$ is complete by Lemma 2.14, the sequences converge to polynomials $F_1$ and $F_2$ with $F = F_1F_2$ and $\bar{F}_1 = f_1$, $\bar{F}_2 = f_2$. This shows existence.

To show uniqueness, assume that $\tilde{F}_1$ and $\tilde{F}_2$ are another solution. Then

$$0 = F - F = F_1F_2 - \tilde{F}_1\tilde{F}_2 = (F_1 - \tilde{F}_1)F_2 + (F_2 - \tilde{F}_2)\tilde{F}_1 .$$

The map $\Phi$ as above, but using $\tilde{F}_1$ and $F_2$, is still invertible (this only uses that the reductions are $f_1$ and $f_2$), which immediately gives $F_1 - \tilde{F}_1 = 0$ and $F_2 - \tilde{F}_2 = 0$. ❑

We draw some conclusions from this.

2.16. **Corollary.** *Let $(K, |\cdot|)$ be a field that is complete with respect to a non-archimedean absolute value, let $R$ be its valuation ring and $k$ its residue field and let $f \in R[x]$ be monic. Assume that $\bar{f} \in k[x]$ has a simple root $a \in k$. Then $f$ has a unique root $\alpha \in R$ such that $\bar{\alpha} = a$.*

**COR** Hensel's Lemma for roots

*Proof.* This is the case $f_1 = x - a$ of Theorem 2.15. Note that the assumption that $a$ is a simple root of $\bar{f}$ implies that the cofactor $f_2 = \bar{f}/(x - a)$ is coprime to $f_1$. ❑

2.17. **Corollary.** *Let $(K, |\cdot|)$ be a field that is complete with respect to a non-archimedean absolute value and let $f = a_0 + a_1x + \ldots + a_nx^n \in K[x]$ with $a_n \ne 0$. Assume that there is $0 < m < n$ with $|a_m| = |f|$ and that $|a_0| < |f|$ or $|a_n| < |f|$. Then $f$ is reducible.*

**COR** reducibility of certain polynomials

*Proof.* After scaling $f$ we can assume that $|f| = 1$. If $|a_0| < 1$, then let $m$ be minimal with $|a_m| = 1$. Then $\bar{f} = x^m f_2$ with $f_2(0) \ne 0$, so that $f_1 = x^m$ and $f_2$ are coprime. By Theorem 2.15 there is a factorization $f = F_1F_2$ with $\deg F_1 = m$. Since $0 < m < n = \deg f$, this shows that $f$ is reducible.

If $|a_n| < 1$, then let $m$ be maximal with $|a_m| = 1$. Then $\bar{f} = f_1 \cdot \bar{a}_m$ with $f_1$ monic and (trivially) coprime to $f_2 = \bar{a}_m$. Again by Theorem 2.15 there is a factorization $f = F_1F_2$ with $\deg F_1 = m$. Since again $0 < m < n = \deg f$, this shows that $f$ is reducible also in this case. ❑

The following example really belongs right after Definition 2.12.

2.18. **Example.** The residue field of $\mathbb{Q}_p$ is $\mathbb{F}_p$. This essentially follows from Lemma 2.10; the details are left as an exercise. ♣

**EXAMPLE** $\mathbb{Q}_p$ has $\mathbb{F}_p$ as residue field

We will now look at field extensions of complete fields with absolute value. First we introduce the norm and trace of an element in a finite field extension.

2.19. **Definition.** Let $K \subset L$ be a finite field extension and let $\alpha \in L$. Then multiplication by $\alpha$ induces a $K$-linear map $m_\alpha \colon L \to L$. We define the *norm* $\mathrm{N}(\alpha)$ and *trace* $\mathrm{Tr}(\alpha)$ of $\alpha$ to be the determinant and trace of $m_\alpha$, respectively. **DEF** norm and trace

If we want to make clear which fields are involved, we write $\mathrm{N}_{L/K}(\alpha)$ and $\mathrm{Tr}_{L/K}(\alpha)$. ♢

Norms and traces have the following properties.

- If $\alpha \in K$, then clearly $\mathrm{N}_{L/K}(\alpha) = \alpha^{[L:K]}$ (since $m_\alpha$ can be taken to be $\alpha I_{[L:K]}$).
- If $K \subset L$ is separable, then we can also write

$$\mathrm{N}(\alpha) = \prod_{\sigma \colon L \hookrightarrow \bar{K}} \sigma(\alpha) \qquad \text{and} \qquad \mathrm{Tr}(\alpha) = \sum_{\sigma \colon L \hookrightarrow \bar{K}} \sigma(\alpha)\,,$$

  where $\sigma$ runs through all embeddings of $L$ into a fixed algebraic closure $\bar{K}$ of $K$. (This is because the $\sigma(\alpha)$ are the eigenvalues of $m_\alpha$.)
- Norms and traces are transitive: if $K \subset L \subset L'$, then for $\alpha \in L'$ we have $\mathrm{N}_{L/K}(\mathrm{N}_{L'/L}(\alpha)) = \mathrm{N}_{L'/K}(\alpha)$ and $\mathrm{Tr}_{L/K}(\mathrm{Tr}_{L'/L}(\alpha)) = \mathrm{Tr}_{L'/K}(\alpha)$.
- The norm is multiplicative and the trace is additive (even $K$-linear); this follows from the corresponding properties of determinants and traces of linear endomorphisms.
- If $L = K(\alpha)$, then the characteristic polynomial of $m_\alpha$ agrees with the minimal polynomial of $\alpha$, so its constant term is $\pm\,\mathrm{N}(\alpha)$.

2.20. **Lemma.** *Let $(K, |\cdot|)$ be a field that is complete with respect to a non-archimedean absolute value and let $R$ be its valuation ring. Let $K \subset L$ be a finite field extension and let $\alpha \in L$ have norm $\mathrm{N}(\alpha) \in R$. Then $\alpha$ is integral over $R$, i.e., $\alpha$ is a root of a monic polynomial with coefficients in $R$.* **LEMMA** integrality

*Proof.* Let $L' = K(\alpha) \subset L$. Then $\mathrm{N}(\alpha) = \mathrm{N}_{L'/K}(\alpha)^{[L:L']}$. Let $f \in K[x]$ be the minimal polynomial of $\alpha$. Its constant term $a_0$ is $\pm\,\mathrm{N}_{L'/K}(\alpha)$ and so the assumption implies $|a_0| \le 1$. If we had $|f| > 1$, then $f$ would be reducible by Corollary 2.17, a contradiction. So $|f| = 1$, meaning that $f \in R[x]$, and $\alpha$ is a root of the monic polynomial $f$. ❑

2.21. **Corollary.** *In the situation of Lemma 2.20 we have $|\,\mathrm{N}(1+\alpha)| \le 1$.* **COR** bound for $\mathrm{N}(1+\alpha)$

*Proof.* We know that $\alpha$ is a root of a monic polynomial $f \in R[x]$. Then $1+\alpha$ is a root of the monic polynomial $f(x-1) \in R[x]$, and since a power of its constant term, which is in $R$, is $\pm\,\mathrm{N}(1+\alpha)$, it follows that $|\,\mathrm{N}(1+\alpha)| \le 1$. ❑

2.22. **Theorem.** *Let $(K, |\cdot|)$ be a complete field with a nontrivial non-archimedean absolute value and let $L$ be an algebraic extension of $K$. Then there is a unique absolute value $|\cdot|'$ on $L$ that extends $|\cdot|$. If the extension $K \subset L$ is finite, then $(L, |\cdot|')$ is complete.* **THM** extensions of complete fields

*Proof.* Since any algebraic extension of $K$ can be obtained as an increasing union of finite extensions, it suffices to consider the finite case. So let $[L : K] = n$. We first show existence. To this end we define $|\alpha|' = |\operatorname{N}(\alpha)|^{1/n}$. Then it is clear that $|\cdot|'$ satisfies properties (1) and (2) of Definition 2.1. To show property (3′), let $\alpha, \beta \in L$ and assume that $|\alpha|' \geq |\beta|'$. Then $|\operatorname{N}(\beta/\alpha)| \leq 1$, hence by Corollary 2.21

$$|\alpha + \beta|'^n = |\operatorname{N}(\alpha + \beta)| = |\operatorname{N}(\alpha)| \cdot |\operatorname{N}(1 + \beta/\alpha)| \leq |\operatorname{N}(\alpha)| = (\max |\alpha|', |\beta|')^n .$$

It is also clear that $|\cdot|'$ agrees with $|\cdot|$ on $K$.

To show uniqueness, let $|\cdot|''$ be another absolute value on $L$ extending $|\cdot|$. Let $\alpha \in L$ with $|\alpha|' \leq 1$, so that $|\operatorname{N}(\alpha)| \leq 1$. Then the minimal polynomial $f$ of $\alpha$ is in $R[x]$. If we had $|\alpha|'' > 1$, then by the ultrametric triangle inequality applied to $f(\alpha) = 0$ we would obtain a contradiction, since then $\alpha^n$ would be the unique term of maximal absolute value, compare Lemma 2.13. So we must have $|\alpha|'' \leq 1$. If $|\alpha|' \geq 1$, then $|\alpha^{-1}|' \leq 1$, so $|\alpha^{-1}|'' \leq 1$ and $|\alpha|'' \geq 1$. This implies that $|\alpha|' < 1 \iff |\alpha|'' < 1$, from which it follows that the two absolute values on $L$ are equivalent (compare Problem (1) on Exercise sheet 1). Since they have the same restriction to $K$, they must then be equal.

It remains to show that $(L, |\cdot|')$ is complete. Let $(b_1, b_2, \ldots, b_m)$ be $K$-linearly independent elements of $L$. We claim that there are constants $c, C > 0$ (depending on $b_1, b_2, \ldots, b_m$) such that

$$c \max\{|a_1|, \ldots, |a_m|\} \leq |a_1 b_1 + \ldots + a_m b_m|' \leq C \max\{|a_1|, \ldots, |a_m|\} \tag{2.1}$$

for all $a_1, a_2, \ldots, a_m \in K$. We prove this by induction on $m$. The cases $m \leq 1$ are trivial. So let $m \geq 2$. The upper bound is easy:

$$\begin{aligned} |a_1 b_1 + \ldots + a_m b_m|' &\leq \max\{|a_1 b_1|', \ldots, |a_m b_m|'\} \\ &\leq \max\{|b_1|', \ldots, |b_m|'\} \cdot \max\{|a_1|, \ldots, |a_m|\} . \end{aligned}$$

To prove the lower bound, we argue by contradiction. If there is no lower bound, then there are $a_1, \ldots, a_m$ with $\max\{|a_1|, \ldots, |a_m|\} = 1$ and $|a_1 b_1 + \ldots + a_m b_m|'$ arbitrarily small. Pick a sequence $(a_1^{(k)}, \ldots, a_m^{(k)})$ of such tuples so that

$$|a_1^{(k)} b_1 + \ldots + a_m^{(k)} b_m|' < 2^{-k} . \tag{2.2}$$

By passing to a sub-sequence and scaling we can assume that $a_j^{(k)} = 1$ for all $k$ and for some $j$, say $j = 1$. Taking differences, we see that

$$\bigl|(a_2^{(k+1)} - a_2^{(k)}) b_2 + \ldots + (a_m^{(k+1)} - a_m^{(k)}) b_m\bigr|' < 2^{-k} \quad \text{for all } k.$$

By induction, there is $c > 0$ such that $|a_2 b_2 + \ldots + a_m b_m|' \geq c \max\{|a_2|, \ldots, |a_m|\}$. This implies that $|a_j^{(k+1)} - a_j^{(k)}| \leq c^{-1} 2^{-k}$ for all $2 \leq j \leq m$, so $(a_j^{(k)})_k$ is a Cauchy sequence and converges to a limit $a_j$. Taking the limit in (2.2) we find that

$$b_1 + a_2 b_2 + \ldots + a_m b_m = 0 ,$$

which contradicts the linear independence of $b_1, b_2, \ldots, b_m$.

Now let $(b_1, b_2, \ldots, b_n)$ be a $K$-basis of $L$, and let $c > 0$ be the associated constant. If $(\alpha_m)$ is a Cauchy sequence in $L$, write $\alpha_m = a_{1m} b_1 + \ldots + a_{nm} b_n$. Then

$$|a_{jm'} - a_{jm}| \leq c^{-1} |\alpha_{m'} - \alpha_m|' ,$$

so each sequence $(a_{jm})_m$ is a Cauchy sequence in $K$ and so converges to some $a_j \in K$. But then $\alpha_m \to a_1 b_1 + \cdots + a_n b_n$ converges as well. ❑

We remark that (2.1) implies that the topology on $L$ induced by $|\cdot|'$ is the same as the topology induced by any $K$-linear isomorphism $L \to K^n$, where $K^n$ has the product topology.

Since $|\cdot|'$ is uniquely determined and extends $|\cdot|$, one simply writes $|\cdot|$ for the absolute value on $L$.

We note that it can be shown that a field that is complete with respect to an archimedean absolute value must be isomorphic to $\mathbb{R}$ or $\mathbb{C}$ (with the usual absolute value) and that a *locally compact* (and then necessarily complete) non-archimedean field of characteristic 0 must be isomorphic to a finite extension of $\mathbb{Q}_p$ for some prime number $p$. (In characteristic $p$, it will be isomorphic to a finite extension of the field $\mathbb{F}_p((t))$ of formal Laurent series over $\mathbb{F}_p$, which is the field of fractions of the ring of formal power series $\mathbb{F}_p[[t]]$.)

2.23. **Corollary.** *Let $\bar{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$. Then $\bar{\mathbb{Q}}_p$ has a unique absolute value extending $|\cdot|_p$ on $\mathbb{Q}_p$. If $\sigma \in \operatorname{Aut}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is any automorphism, then $|\sigma(\alpha)|_p = |\alpha|_p$ for all $\alpha \in \bar{\mathbb{Q}}_p$.*

**COR**
$\bar{\mathbb{Q}}_p$ has unique abs. value

*Proof.* $\bar{\mathbb{Q}}_p$ is an algebraic extension of $\mathbb{Q}_p$, so Theorem 2.22 applies. The second statement follows from the uniqueness of the absolute value, since $\alpha \mapsto |\sigma(\alpha)|_p$ is another absolute value on $\bar{\mathbb{Q}}_p$ extending the $p$-adic absolute value on $\mathbb{Q}_p$. ❑

Next we want to show that $\bar{\mathbb{Q}}_p$ is *not* complete (contrary to the algebraic closure $\mathbb{C}$ of the completion $\mathbb{R}$ of $\mathbb{Q}$ with respect to the usual absolute value).

2.24. **Lemma.** *$\bar{\mathbb{Q}}_p$ is not complete with respect to $|\cdot|_p$.*

**LEMMA**
$\bar{\mathbb{Q}}_p$ not complete

*Proof.* For $n \geq 1$, let $\zeta_n \in \bar{\mathbb{Q}}_p$ be a primitive $(p^{2^n} - 1)$-th root of unity. It follows from properties of finite fields that $\zeta_n$ is a root of a monic polynomial $f_n \in \mathbb{Z}_p[x]$ of degree $2^n$ that reduces to an irreducible polynomial in $\mathbb{F}_p[x]$; in particular, $f_n$ is irreducible itself. Since $\zeta_n$ is a power of $\zeta_{n+1}$, we obtain a tower of fields

$$\mathbb{Q}_p = \mathbb{Q}_p(\zeta_1) \subset \mathbb{Q}_p(\zeta_2) \subset \mathbb{Q}_p(\zeta_3) \subset \ldots \subset \bar{\mathbb{Q}}_p$$

such that $[\mathbb{Q}_p(\zeta_{n+1}) : \mathbb{Q}_p(\zeta_n)] = 2$ for all $n \geq 1$. We now consider the series $\sum_{n=1}^{\infty} \zeta_n p^n$. If $\bar{\mathbb{Q}}_p$ were complete, then series would converge (since $|\zeta_n|_p = 1$). So we will show that the series does *not* converge in $\bar{\mathbb{Q}}_p$. The proof will be by contradiction. So we assume that the series has a limit $\alpha \in \bar{\mathbb{Q}}_p$. Then $m := [\mathbb{Q}_p(\alpha) : \mathbb{Q}_p]$ is finite. Let $\sigma \in \operatorname{Aut}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. By Corollary 2.23, $\sigma$ is continuous with respect to the $p$-adic topology. This implies that

$$\sigma(\alpha) = \sum_{n=2}^{\infty} \sigma(\zeta_n) p^n \,.$$

Since $[\mathbb{Q}_p(\alpha) : \mathbb{Q}_p] = m$, there are exactly $m$ values that $\sigma(\alpha)$ can take (namely, the roots of the minimal polynomial of $\alpha$ over $\mathbb{Q}_p$). Now pick some $n$ such that $2^n > m$. Note that $s_n = \sum_{k=1}^{n} \zeta_k p^k$ generates the field $\mathbb{Q}_p(\zeta_n)$ (this is easily seen by induction). So, as $\sigma$ runs through the automorphisms of $\bar{\mathbb{Q}}_p$ over $\mathbb{Q}_p$, $\sigma(s_n)$ takes exactly $2^n$ distinct values. Since the various possible values of $\sigma(\zeta_n)$ all differ mod $p$ (this is because $f_n$ has only simple roots in $\bar{\mathbb{F}}_p$), we conclude that the various values of $\sigma(s_n)$ all differ mod $p^{n+1}$. On the other hand,

$$\sigma(\alpha) \equiv \sigma(s_n) \bmod p^{n+1} \,,$$

and there are only $m < 2^n$ possibilities for the left hand side. This gives the desired contradiction. ❑

2.25. **Definition.** We define $\mathbb{C}_p$ to be the completion of $\bar{\mathbb{Q}}_p$. ♢ **DEF** $\mathbb{C}_p$

2.26. **Lemma.** *Let $(K, |\cdot|)$ be an algebraically closed field with a non-archimedean absolute value, and let $K'$ be its completion. Then $K'$ is also algebraically closed.* **LEMMA** completion stays alg. closed

*Proof.* Let $K' \subset L$ be a finite field extension and let $\alpha \in L$; after scaling $\alpha$ by an element of $K$, we can assume that $|\alpha| \leq 1$ (recall that $L$ has a unique absolute value extending that of $K'$). We must show that $\alpha \in K'$. Let $f \in K'[x]$ be the minimal polynomial of $\alpha$. Since $K$ is dense in $K'$, there is a sequence $(f_n)$ of monic polynomials in $K[x]$ such that $|f_n - f| < 2^{-n}$. Since $K$ is algebraically closed, each $f_n$ splits into linear factors. By the ultrametric triangle inequality (and using $|\alpha| \leq 1$), we have

$$\prod_{\alpha' \colon f_n(\alpha')=0} |\alpha - \alpha'| = |f_n(\alpha)| = |f_n(\alpha) - f(\alpha)| \leq |f_n - f| < 2^{-n} .$$

There must then be a root $\alpha_n$ of $f_n$ such that $|\alpha - \alpha_n| < 2^{-n/\deg(f)}$. This implies that $(\alpha_n)$ converges in $L$ to $\alpha$. But $K'$ is the closure of $K$, therefore the limit $\alpha$ must already be in $K'$. ❑

So $\mathbb{C}_p$ is the unique (up to isomorphism) minimal extension of $\mathbb{Q}_p$ that is complete and algebraically closed. So in this sense, $\mathbb{C}_p$ is the $p$-adic analogue of the field of complex numbers.

The residue field of $\bar{\mathbb{Q}}_p$ and of $\mathbb{C}_p$ is $\bar{\mathbb{F}}_p$. Since this is infinite, it follows that neither of these two fields is locally compact: the closed ball around zero of radius 1 contains infinitely many elements whose pairwise distance is 1 (a system of representatives of the residue classes), and so no sub-sequence of a sequence of such elements can converge.

## 3. Newton Polygons

Let $(K, |\cdot|)$ be a complete field with a nontrivial non-archimedean absolute value. Consider a polynomial $0 \neq f \in K[x]$. Let $\alpha \in \bar{K}$ be a root of $f$. We have seen that there is a unique extension of $|\cdot|$ to $\bar{K}$, so it makes sense to consider $|\alpha|$. We will now describe a method that determines the absolute values of the roots of $f$ and how often they occur.

In this context it is advantageous to switch to a 'logarithmic' version of the absolute value. We fix a positive real number $c$ and set

$$v(x) = -c \log |x| \quad \text{for } x \in K^\times \quad \text{and} \quad v(0) = +\infty\,.$$

3.1. **Definition.** The map $v\colon K \to \mathbb{R} \cup \{+\infty\}$ is the *(additive) valuation* associated to $|\cdot|$. ♢ **DEF** valuation

Of course, changing $c$ will scale $v$ by a positive factor, so $v$ is not uniquely determined.

Corresponding to the properties of absolute values, the valuation satisfies

(1) $v(x) < \infty$ if $x \in K^\times$;

(2) $v(xy) = v(x) + v(y)$ for all $x, y \in K$;

(3) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$, with equality when $v(x) \neq v(y)$.

The adjective 'additive' refers to the second property.

When dealing with $p$-adic fields like $\mathbb{Q}_p$, $\bar{\mathbb{Q}}_p$ or $\mathbb{C}_p$, we choose $c = 1/\log p$; then $v = v_p$ on $\mathbb{Q}_p$, so $v(\mathbb{Q}_p^\times) = \mathbb{Z}$ and (as is easily seen) $v(\bar{\mathbb{Q}}_p^\times) = v(\mathbb{C}_p^\times) = \mathbb{Q}$.

Now let $\alpha \in \bar{K}$ be a root of $f = a_0 + a_1 x + \ldots + a_n x^n \in K[x]$ with $a_n \neq 0$. Then

$$a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_n\alpha^n = 0\,,$$

so by Lemma 2.13 there must be (at least) two terms in the sum whose absolute value is maximal, or equivalently, whose valuation is minimal. These valuations are

$$v(a_0),\ v(a_1) + v(\alpha),\ v(a_2) + 2v(\alpha),\ \ldots,\ v(a_n) + nv(\alpha)\,.$$

So we need to have $0 \leq k < m \leq n$ such that

$$v(a_k) + kv(\alpha) = v(a_m) + mv(\alpha) \leq v(a_j) + jv(\alpha) \quad \text{for all } 0 \leq j \leq n.$$

This is equivalent to saying that the line of slope $-v(\alpha)$ through the points $(k, v(a_k))$ and $(m, v(a_m))$ in the plane has no point $(j, v(a_j))$ below it. This prompts the following definition.

3.2. **Definition.** Let $0 \neq f = a_0 + a_1 x + \ldots + a_n x^n \in K[x]$ with $K$ as above. The *Newton polygon* of $f$ is the lower convex hull of the set of the points $(j, v(a_j))$ for $0 \leq j \leq n$ such that $a_j \neq 0$, i.e., the union of all line segments joining two of these points and such that the line through these points does not run strictly above any of the other points. A maximal such line segment is a *segment* of the Newton polygon; it has a *slope* (which is just its usual slope) and a *length*, which is the length of its projection to the $x$-axis. ♢ **DEF** Newton polygon

3.3. **Example.** Consider the polynomial

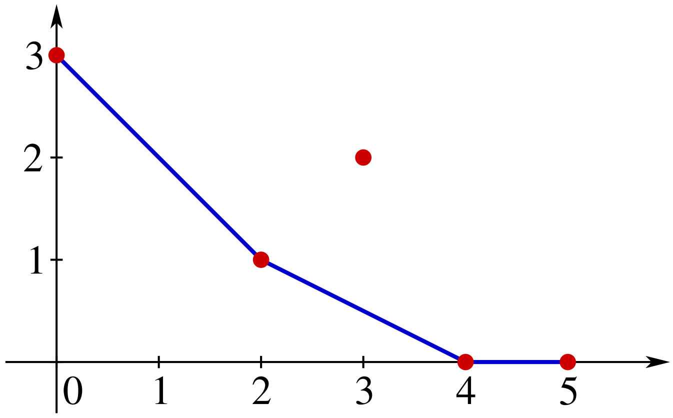
**EXAMPLE** Newton polygon

$$f = x^5 + 3x^4 + 4x^3 + 6x^2 + 8 \in \mathbb{Q}_2[x] .$$

The points $(j, v(a_j))$ with $a_j \neq 0$ are

$$(0,3),\ (2,1),\ (3,2),\ (4,0),\ (5,0) .$$

We find three segments, forming a broken line with vertices $(0,3)$, $(2,1)$, $(4,0)$, $(5,0)$. The slopes are $-1$, $-1/2$ and $0$, and the lengths are $2$, $2$ and $1$.




♣

What we did above amounts to the statement that the valuations of the roots of $f$ are among the slopes taken negatively of the Newton polygon. We now want to prove a converse and give a more precise statement. We first introduce a variation of the absolute value on the polynomial ring.

3.4. **Definition.** Let $(K, |\cdot|)$ be a field with a nontrivial non-archimedean absolute value and let $r > 0$. For $f = a_0 + a_1 x + \ldots + a_n x^n \in K[x]$ we define

**DEF** $|\cdot|_r$ on $K[x]$

$$|f|_r := \max\{|a_j| r^j : 0 \le j \le n\} .$$

When $f \neq 0$, then we set

$$\ell_r(f) = \max\{j : |a_j| r^j = |f|_r\} - \min\{j : |a_j| r^j = |f|_r\} \in \mathbb{Z}_{\ge 0} . \qquad \diamondsuit$$

We observe that $\ell_r(f)$ is strictly positive if and only if $c \log r$ is the slope of a segment of the Newton polygon of $f$, and that in this case $\ell_r(f)$ is the length of the corresponding segment.

One can easily adapt the proof for the case $r = 1$ to show that $|\cdot|_r$ is an absolute value on $K[x]$ (in the sense that it satisfies properties (1), (2) and (3′) of Definition 2.1). It also restricts to $|\cdot|$ on $K$.

3.5. **Lemma.** *Let $f, g \in K[x]$ be nonzero polynomials and let $r > 0$. Then we have*

**LEMMA** additivity of $\ell_r$

$$\ell_r(fg) = \ell_r(f) + \ell_r(g) .$$

*Proof.* If $f = a_0 + a_1 x + \ldots + a_n x^n$, write

$$n_-(f) = \min\{j : |a_j| r^j = |f|_r\} \qquad \text{and} \qquad n_+(f) = \max\{j : |a_j| r^j = |f|_r\}$$

and similarly for $g$ and $fg$. Precisely as in the proof of property (2) for $|\cdot|_r$ (i.e., basically Gauss' Lemma) one sees that

$$n_-(fg) = n_-(f) + n_-(g) \qquad \text{and} \qquad n_+(fg) = n_+(f) + n_+(g) ,$$

which implies the claim, since $\ell_r(f) = n_+(f) - n_-(f)$ and similarly for $g$ and $fg$. ❑

3.6. **Theorem.** *Let $0 \neq f \in K[x]$ and let $r > 0$. Then the number of roots $\alpha \in \bar{K}$ of $f$ (counted with multiplicity) such that $|\alpha| = r$ is exactly $\ell_r(f)$.*

**THM** roots and Newton polygon

In terms of the Newton polygon, this says that $f$ has roots of valuation $s$ if and only if its Newton polygon has a segment of slope $-s$, and the number of such roots (counted with multiplicity) is exactly the length of the segment.

*Proof.* We can assume that $K = \bar{K}$. The proof is by induction on the degree of $f$. If $f$ is constant, there is nothing to show. If $f = x - \alpha$, then $\ell_r(f) = 0$ if $|\alpha| \neq r$ and $\ell_r(f) = 1$ if $|\alpha| = r$.

Now assume $f$ is not constant and let $\beta \in \bar{K}$ be a root of $f$. Then $f = (x - \beta)f_1$ for some $0 \neq f_1 \in K[x]$. By the inductive hypothesis, the number of roots $\alpha$ of $f_1$ such that $|\alpha| = r$ is $\ell_r(f_1)$. If $|\beta| = r$, then the number of such roots of $f$ is $\ell_r(f_1) + 1 = \ell_r(f_1) + \ell_r(x - \beta) = \ell_r(f)$ (using Lemma 3.5). If $|\beta| \neq r$, then the number of such roots of $f$ is $\ell_r(f_1) = \ell_r(f_1) + \ell_r(x - \beta) = \ell_r(f)$ again. ❑

3.7. **Lemma.** *Let $f \in K[x]$ be irreducible. Then the Newton polygon of $f$ consists of a single segment.*

**LEMMA** Newton polygon of irreducible polynomial

*Proof.* We assume for simplicity that $K$ has characteristic zero. Then $\bar{K}$ is separable over $K$ and the roots of $f$ form one orbit under the Galois group $\mathrm{Aut}(\bar{K}/K)$. Since the absolute value is invariant under the action of this group, we see that all roots of $f$ have the same absolute value. The claim then follows from Theorem 3.6.

(In characteristic $p$, let $\alpha$ be a root of $f$. Then for some $n \geq 0$, $\alpha^{p^n}$ is separable over $K$ and $f = g(x^{p^n})$ for an irreducible polynomial $g \in K[x]$. The previous argument applies to $g$, but this then implies the claim also for $f$.) ❑

This leads to the following consequence.

3.8. **Lemma.** *Let $0 \neq f \in K[x]$ with $f(0) \neq 0$ and let $\sigma_1, \ldots, \sigma_m$ be the segments of the Newton polygon of $f$. Then there is a factorization*

**LEMMA** slope factorization

$$f = f_1 f_2 \cdots f_m$$

*such that the Newton polygon of $f_j$ is a single segment with the same slope and length as $\sigma_j$.*

*Proof.* We can assume that $f$ is monic. Let $f = h_1 \cdots h_n$ be the factorization of $f$ into monic irreducible polynomials over $K$. By Lemma 3.7, the Newton polygon of each $h_j$ consists of a single segment. Let $s_1, s_2, \ldots, s_m$ be the distinct slopes that occur for these segments; we can number them so that $s_j$ is the slope of $\sigma_j$. We then define $f_j$ to be the product of the $h_i$ whose slope is $s_j$. The claim follows from the additivity of the lengths of segments of the same slope, Lemma 3.5. ❑

We extend $|\cdot|_r$ to an absolute value on the field $K(x)$ of rational functions in one variable over $K$ in the usual way. For a given rational function $f$, we can then study how $|f|_r$ varies with $r$.

3.9. **Theorem.** *Let $f \in K(x)^\times$. Then the function*

$$\varphi_f \colon \mathbb{R} \longrightarrow \mathbb{R}\,, \qquad s \longmapsto -c \log |f|_{e^{-s/c}}$$

*is piecewise affine with integral slopes. If $\partial_- \varphi_f(s) - \partial_+ \varphi_f(s) = \nu$, then $\nu$ is the number of zeros $\alpha$ of $f$ with $v(\alpha) = s$ minus the number of poles $\alpha$ of $f$ with $v(\alpha) = s$ (each counted with multiplicity).*

**THM**
variation of $|f|_r$ with $r$

Here

$$\partial_+ \varphi_f(s) = \lim_{\varepsilon \searrow 0} \frac{\varphi_f(s+\varepsilon) - \varphi_f(s)}{\varepsilon} \quad \text{and} \quad \partial_- \varphi_f(s) = \lim_{\varepsilon \searrow 0} \frac{\varphi_f(s) - \varphi_f(s-\varepsilon)}{\varepsilon}$$

are the *right* and *left derivatives* of the piecewise affine function $\varphi_f$.

*Proof.* We can write $f = \gamma \prod_{j=1}^n (x - \alpha_j)^{e_j}$ for some $\gamma \in K$, $\alpha_j \in \bar{K}$ and $e_j \in \mathbb{Z}$, so

$$\varphi_f(s) = -c \log |\gamma| - c \sum_{j=1}^n e_j \log |x - \alpha_j|_{e^{-s/c}} = v(\gamma) + \sum_{j=1}^n e_j \varphi_{x-\alpha_j}(s)\,,$$

and it suffices to prove the claim for $f = x - \alpha$. In this case, we have

$$\varphi_{x-\alpha}(s) = -c \log \max\{e^{-s/c}, |\alpha|\} = \min\{s, v(\alpha)\}$$

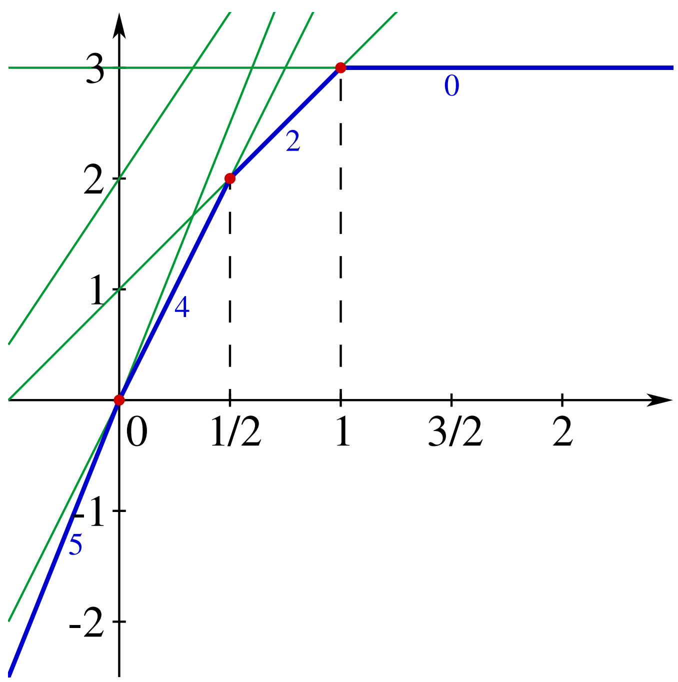
(recall that $v(0) = +\infty$). This is a piecewise affine function with slopes 1 and 0 (unless $\alpha = 0$). If $\alpha = 0$, the slope is constant, so $\nu = 0$ for all $s$, and there are no zeros or poles $\alpha$ with $v(\alpha) = s$. If $\alpha \neq 0$, then the slope changes from 1 to 0 at $s = v(\alpha)$, so $\nu = 1$ there and $\nu = 0$ for all other $s$. ❑

3.10. **Example.** For the polynomial

$$f = x^5 + 3x^4 + 4x^3 + 6x^2 + 8 \in \mathbb{Q}_2[x]$$

from Example 3.3, the graph of $\varphi_f$ looks as follows.

**EXAMPLE**
$\varphi_f$ for $f$ from Example 3.3



The green lines are the graphs of $s \mapsto v(a_j) + js$ (for $j = 0, 2, 3, 4, 5$); $\varphi_f$ is the minimum of these functions. The segments of the graph of $\varphi_f$ correspond to the vertices of the Newton polygon of $f$ and the vertices of the graph of $\varphi_f$ correspond to the segments of the Newton polygon of $f$. ♣

We give another characterization of $|f|_r$.

3.11. **Lemma.** *Let $(K, |\cdot|)$ be a complete non-archimedean field, let $r > 0$ and $0 \neq f \in K[x]$. Then*

**LEMMA** $|\cdot|_r$ as supremum norm

$$|f|_r = \sup\{|f(\alpha)| : \alpha \in \bar{K}, |\alpha| \le r\}\,.$$

*Proof.* Let $f = a_0 + a_1 x + \ldots + a_n x^n$. If $|\alpha| \le r$, then

$$|f(\alpha)| = |a_0 + a_1\alpha + \ldots + a_n\alpha^n| \le \max\{|a_0|, |a_1|r, \ldots, |a_n|r^n\} = |f|_r\,.$$

Conversely, for any $\varepsilon > 0$ there is $r - \varepsilon < \rho \le r$ such that $\rho \in |\bar{K}^\times|$ and $c \log \rho$ is not a slope of the Newton polygon of $f$ (recall that $|\bar{K}^\times|$ is dense in $\mathbb{R}_{>0}$; there are only finitely many slopes). Then for any $\alpha \in \bar{K}$ such that $|\alpha| = \rho$, there is a unique term $a_j\alpha^j$ of maximal absolute value and therefore we have $|f(\alpha)| = |f|_\rho$. Letting $\rho$ tend to $r$, we find $|f|_r \le \sup\{|f(\alpha)| : \alpha \in \bar{K}, |\alpha| \le r\}$. ❑

## 4. Multiplicative seminorms and Berkovich spaces

The absolute values $|\cdot|_r$ on the polynomial ring $K[x]$ that we have studied in the last section are examples of 'multiplicative seminorms'.

**4.1. Definition.** Let $K$ be a field with absolute value $|\cdot|$ and let $A$ be a $K$-algebra (i.e., $A$ is a ring with a ring homomorphism $K \to A$ that gives $A$ a compatible structure as a $K$-vector space). A *multiplicative seminorm* on $A$ is a map

**DEF** multiplicative seminorm

Banach algebra

$$A \longrightarrow \mathbb{R}_{\geq 0}, \qquad a \longmapsto \|a\|$$

such that

(1) $\|\cdot\|$ restricts to $|\cdot|$ on $K$;

(2) $\|a+b\| \leq \|a\| + \|b\|$ for all $a, b \in A$;

(3) $\|ab\| = \|a\| \cdot \|b\|$ for all $a, b \in A$.

If in addition $a = 0$ is the only element with $\|a\| = 0$, then $\|\cdot\|$ is a *multiplicative norm* (which is the same as an absolute value on $A$ extending $|\cdot|$). In general, we call $\ker \|\cdot\| = \{a \in A : \|a\| = 0\}$ the *kernel* of $\|\cdot\|$; it is a prime ideal in $A$.

A $K$-algebra $A$ with a fixed multiplicative norm such that $A$ is complete with respect to this norm is a *Banach algebra* over $K$. ♢

A *seminorm* on $A$ only needs to satisfy $\|ab\| \leq \|a\| \cdot \|b\|$ with equality for $a \in K$; similarly for a *norm*.

If $|\cdot|$ is non-archimedean, then $\|\cdot\|$ also satisfies the ultrametric triangle inequality, since then

$$\begin{aligned}\|a+b\|^n = \|(a+b)^n\| &= \Big\|\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}\Big\| \\ &\leq \sum_{j=0}^{n}\Big|\binom{n}{j}\Big|\|a\|^j\|b\|^{n-j} \leq \sum_{j=0}^{n}\|a\|^j\|b\|^{n-j} \leq (n+1)(\max\{\|a\|,\|b\|\})^n\end{aligned}$$

and $\sqrt[n]{n+1} \to 1$ as $n \to \infty$.

We introduce the following notation.

**4.2. Definition.** Let $(K, |\cdot|)$ be a field with absolute value, let $a \in K$ and $r \geq 0$. Then

**DEF** $D(a,r)$

$$D(a,r) := D_K(a,r) := \{\xi \in K : |\xi - a| \leq r\}$$

is the closed disk of radius $r$ around $a$. ♢

**4.3. Examples.** Let $(K, |\cdot|)$ be a complete non-archimedean field.

**EXAMPLES** multiplicative seminorms on $K[x]$

(1) For any $a \in K$, the map

$$f \longmapsto \|f\|_{a,0} := |f(a)|$$

is a multiplicative seminorm on $K[x]$.

(2) For any $a \in K$ and any $r > 0$, the map

$$f \longmapsto \|f\|_{a,r} := |f(x+a)|_r$$

is a multiplicative (semi)norm on $K[x]$.

(3) Let $(a_n)$ be a sequence in $K$ and $(r_n)$ a strictly decreasing sequence in $\mathbb{R}_{>0}$ such that $D(a_{n+1}, r_{n+1}) \subset D(a_n, r_n)$ for all $n$. Then

$$f \longmapsto \|f\| = \lim_{n\to\infty} \|f\|_{a_n,r_n}$$

is a multiplicative seminorm on $K[x]$.

(1) is clear and (2) follows from the properties of $|\cdot|_r$. For (3) note that by Lemma 3.11,

$$\begin{aligned}\|f\|_{a_{n+1},r_{n+1}} &= \sup\{|f(\alpha)| : \alpha \in \bar{K}, |\alpha - a_{n+1}| \le r_{n+1}\} \\ &\le \sup\{|f(\alpha)| : \alpha \in \bar{K}, |\alpha - a_n| \le r_n\} = \|f\|_{a_n,r_n},\end{aligned}$$

so that the sequence $(\|f\|_{a_n,r_n})$ decreases, hence must have a limit. Properties (2) and (3) from Definition 4.1 then follow by taking the limit in the corresponding relations for the $\|\cdot\|_{a_n,r_n}$. ♣

In fact, this is essentially the full story, at least when $K$ is algebraically closed and complete.

4.4. **Theorem.** *Let $(K, |\cdot|)$ be a complete and algebraically closed non-archimedean field and let $\|\cdot\|$ be a multiplicative seminorm on $K[x]$. Then there is a decreasing nested sequence of disks $D(a_n, r_n)$ such that* **THM** classification of mult. seminorms on $K[x]$

$$\|f\| = \lim_{n\to\infty} \|f\|_{a_n,r_n}$$

*for all $f \in K[x]$.*

*Proof.* Let $\mathcal{D} = \{D(a, r) : a \in K, r > 0, \|\cdot\| \le \|\cdot\|_{a,r}\}$ be the set of all closed disks such that $\|\cdot\|$ is bounded above by the corresponding seminorm. Then for all $a \in K$ we have $D(a, \|x - a\|) \in \mathcal{D}$: if $f(x + a) = a_0 + a_1x + \ldots + a_nx^n$, then

$$\begin{aligned}\|f\| &= \|a_0 + a_1(x - a) + \ldots + a_n(x - a)^n\| \\ &\le \max\{|a_0|, |a_1|\|x - a\|, \ldots, |a_n|\|x - a\|^n\} \\ &= |f(x + a)|_{\|x-a\|} = \|f\|_{a,\|x-a\|}.\end{aligned}$$

In particular, $\mathcal{D}$ is non-empty. Conversely, we have $|x - a|_{a,r} = r$, which implies that if $D(a, r) \in \mathcal{D}$, then $r \ge \|x - a\|$. So

$$\mathcal{D} = \{D(a, r) : a \in K, r \ge \|x - a\|\}.$$

We claim that $\mathcal{D}$ does not contain two disjoint disks: assume that $D_1 = D(a_1, r_1)$ and $D_2 = D(a_2, r_2)$ are in $\mathcal{D}$. Then we have $\|x - a_1\| \le r_1$ and $\|x - a_2\| \le r_2$, so that

$$|a_1 - a_2| = \|(x - a_2) - (x - a_1)\| \le \max\{r_1, r_2\}.$$

If $r_1 \le r_2$, then this implies that $a_1 \in D_2$ and then that $D_1 \subset D_2$; if $r_2 \le r_1$, then we see in the same way that $D_2 \subset D_1$.

Now let $\rho = \inf\{r > 0 : \exists a \in K : D(a, r) \in \mathcal{D}\}$ and choose sequences $(a_n)$ in $K$ and $(r_n)$ in $\mathbb{R}_{>0}$ such that $(r_n)$ is strictly decreasing with $r_n \to \rho$ and $D(a_n, r_n) \in \mathcal{D}$ for all $n$. Then we have $D(a_{n+1}, r_{n+1}) \subset D(a_n, r_n)$. This implies

$$\|f\| \le \lim_{n\to\infty} \|f\|_{a_n,r_n}.$$

We still have to show the reverse inequality. Since every nonzero $f$ is a product of a constant and polynomials of the form $x - a$, it suffices to prove this for the latter. If $|a - a_{n_0}| > r_{n_0}$ for some $n_0$, then

$$\begin{aligned}\|x - a\| &= \|(x - a_{n_0}) - (a - a_{n_0})\| = |a - a_{n_0}| \\ &= \lim_{n\to\infty} \max\{|a - a_n|, r_n\} = \lim_{n\to\infty} \|x - a\|_{a_n,r_n} ,\end{aligned}$$

since $\|x - a_{n_0}\| \le r_{n_0} < |a - a_{n_0}|$. Otherwise, we have $|a - a_n| \le r_n$ for all $n$, which says that $a \in \bigcap_n D(a_n, r_n)$, so that $D(a_n, r_n) = D(a, r_n)$ and we have

$$\lim_{n\to\infty} \|x - a\|_{a_n,r_n} = \lim_{n\to\infty} \|x - a\|_{a,r_n} = \|x - a\|_{a,\rho} = \rho .$$

Since $D(a, \|x - a\|) \in \mathcal{D}$ and $\rho$ is the smallest possible radius of a disk in $\mathcal{D}$, we get $\|x - a\| \ge \rho$ as required. ❑

Note that if $a \in \bigcap_n D(a_n, r_n)$ in the proof above, then $D(a, \rho) \in \mathcal{D}$, since

$$\|f\| \le \lim_{n\to\infty} \|f\|_{a_n,r_n} = \lim_{n\to\infty} \|f\|_{a,r_n} = \|f\|_{a,\rho}$$

for all $f \in K[x]$. Since $\rho \le r$ for any radius $r$ of a disk in $\mathcal{D}$ and no two disks in $\mathcal{D}$ are disjoint, it follows that $D(a, \rho) = \bigcap \mathcal{D}$. We can therefore distinguish the following four types of multiplicative seminorms on $K[x]$.

4.5. **Definition.** Let $\|\cdot\|$ be a multiplicative seminorm on $K[x]$, where $K$ is a complete and algebraically closed non-archimedean field. Let $\mathcal{D}$ be as in the proof of Theorem 4.4.

**DEF** types of mult. seminorms

(1) $\|\cdot\|$ is of *type 1* if $\bigcap \mathcal{D} = \{a\}$ for some $a \in K$. Then $\|f\| = |f(a)|$ and $\ker \|\cdot\|$ is the kernel of the evaluation map $f \mapsto f(a)$.
(2) $\|\cdot\|$ is of *type 2* if $\bigcap \mathcal{D} = D(a, r)$ for some $a \in K$ and $r > 0$ such that $r \in |K^\times|$. Then $\|f\| = |f|_{a,r}$.
(3) $\|\cdot\|$ is of *type 3* if $\bigcap \mathcal{D} = D(a, r)$ for some $a \in K$ and $r > 0$ such that $r \notin |K^\times|$. Then $\|f\| = |f|_{a,r}$.
(4) Finally, $\|\cdot\|$ is of *type 4* if $\bigcap \mathcal{D} = \emptyset$.

In the last three cases, $\|\cdot\|$ is actually a norm. ♢

We note that if $\rho = 0$ in the proof of Theorem 4.4, then the completeness of $K$ implies that $\mathcal{D} = \{a\}$ for some $a \in K$. If $\bigcap \mathcal{D} \ne \emptyset$ for every decreasing nested sequence of disks in $K$, then $K$ is said to be *spherically complete*. (Then no multiplicative seminorms of type 4 exist.) For example, $\mathbb{Q}_p$ is spherically complete, but $\mathbb{C}_p$ is not (Exercise).

**DEF** spherically complete

For types 2, 3 and 4, $\|\cdot\|$ defines an absolute value on $K[x]$. We extend it to an absolute value on the field of fractions $K(x)$, which we can then complete to obtain $\mathcal{H}_{\|\cdot\|}$, and we obtain a $K$-algebra homomorphism $K[x] \to \mathcal{H}_{\|\cdot\|}$. For example, if $\|\cdot\| = |\cdot|_r$ is of type 2 or 3 with $a = 0$, then the corresponding completion of $K[x]$ is the ring

$$K\langle r^{-1}x\rangle := \Big\{\sum_{j=0}^{\infty} a_j x^j \in K[\![X]\!] : \lim_{j\to\infty} |a_j| r^j = 0\Big\}$$

of power series converging on $D(0, r)$ (Exercise), so $\mathcal{H}_{|\cdot|_r}$ is the field of fractions of $K\langle r^{-1}x\rangle$.

If $\|\cdot\|$ is of type 1, then the evaluation map at $a$ gives us a $K$-algebra homomorphism $K[x] \to K =: \mathcal{H}_{\|\cdot\|}$. In each case, we obtain $\|\cdot\|$ as the pull-back of the absolute value of a complete field $\mathcal{H}$ that is a $K$-Banach algebra via a $K$-algebra homomorphism $K[x] \to \mathcal{H}$. Conversely, if we have such a homomorphism $K[x] \to \mathcal{H}$, then pulling back the absolute value of $\mathcal{H}$ to $K[x]$ will give us a multiplicative seminorm on $K[x]$.

4.6. **Definition.** Let $K$ be a complete and algebraically closed non-archimedean field and let $A$ be a finitely generated $K$-algebra, so that $A$ is the coordinate ring of the affine $K$-variety $X = \operatorname{Spec} A$. Then the *Berkovich space* associated to $A$ or $X$ is **DEF** Berkovich space

$$\operatorname{Berk} A := X^{\mathrm{an}} := \{\|\cdot\| \colon A \to \mathbb{R} \text{ multiplicative seminorm on } A\},$$

the set of multiplicative seminorms on $A$. The topology on $X^{\mathrm{an}}$ is the weakest topology that makes the maps $X^{\mathrm{an}} \to \mathbb{R}$, $\|\cdot\| \mapsto \|f\|$, continuous for all $f \in A$. (Concretely, this means that any open set is a union of finite intersections of sets of the form $U_{f,a,b} = \{\|\cdot\| \in X^{\mathrm{an}} : a < \|f\| < b\}$.) ♢

We will usually call elements of $X^{\mathrm{an}}$ *points* and denote them by $\xi$ or similar. In this context, the corresponding multiplicative seminorm will be written $\|\cdot\|_\xi$ and the $K$-Banach algebra obtained by completion, $\mathcal{H}_\xi$.

4.7. **Lemma.** *The topological space $X^{\mathrm{an}}$ as defined in Definition 4.6 is Hausdorff.* **LEMMA** $X^{\mathrm{an}}$ is Hausdorff

*Proof.* Let $\xi, \xi' \in X^{\mathrm{an}}$ be distinct. We must show that there are disjoint open sets $U, U' \subset X^{\mathrm{an}}$ with $\xi \in U$ and $\xi' \in U'$. Since $\xi \neq \xi'$, there must be $f \in A$ such that $\|f\|_\xi \neq \|f\|_{\xi'}$. Assume without loss of generality that $\|f\|_\xi < \|f\|_{\xi'}$ and let $\|f\|_\xi < a < \|f\|_{\xi'}$. Then we can take $U = U_{f,-\infty,a}$ and $U' = U_{f,a,\infty}$. ❑

Note that we can alternatively define $\operatorname{Berk} A$ as the set of all $K$-algebra homomorphisms $A \to \mathcal{H}$ into complete $K$-Banach algebras that are fields, up to an obvious equivalence. The topology is then that of pointwise convergence. This can be seen as similar to the definition of $\operatorname{Spec} A$ as the set of all $K$-algebra homomorphisms into fields, up to an obvious equivalence. Here the topology is again that of pointwise convergence, but with the cofinal topology on the target fields (so that basic open sets are defined by relations $f \neq 0$). Since a Banach algebra has more structure than just a $K$-algebra, there are fewer equivalences and therefore more points in $\operatorname{Berk} A$ than in $\operatorname{Spec} A$. This is made precise by the following statements.

4.8. **Lemma.** *Let $K$, $A$ and $X$ be as in Definition 4.6. Then there is a canonical inclusion of $X(K)$ into $X^{\mathrm{an}}$. The inclusion is continuous when $X(K)$ is given the topology induced by the absolute value on $K$.* **LEMMA** $X(K) \hookrightarrow X^{\mathrm{an}}$

*Proof.* Let $\xi \in X(K)$. Then $\xi$ gives rise to a $K$-algebra homomorphism $A \to K$, $f \mapsto f(\xi)$. We define $\tilde{\xi} \in X^{\mathrm{an}}$ to correspond to the multiplicative seminorm $f \mapsto |f(\xi)|$. Its kernel is the ideal of $A$ consisting of functions vanishing at $\xi$ and so determines $\xi$. This gives us the desired inclusion.

To show that $\xi \mapsto \tilde{\xi}$ is continuous, consider a basic open set $U_{f,a,b}$ in $X^{\mathrm{an}}$. Its preimage is $\{\xi \in X(K) : a < |f(\xi)| < b\}$. Since every $f \in A$ is continuous as a map $X(K) \to K$ (with respect to the $K$-topology) and $|\cdot| \colon K \to \mathbb{R}$ is also continuous, this set is open. ❑

It is in fact the case that the image of $X(K)$ in $X^{\mathrm{an}}$ is dense. We will see this later when $X$ is the affine line.

There is also a natural map in the other direction.

4.9. **Lemma.** *Let $K$, $A$ and $X$ be as in Definition 4.6. Then there is a canonical continuous map* $\operatorname{Berk} A \to \operatorname{Spec} A$ *(where* $\operatorname{Spec} A$ *has the Zariski topology).* **LEMMA** $\operatorname{Berk} A \to \operatorname{Spec} A$

*Proof.* Let $\xi \in \operatorname{Berk} A$. Then $\|\cdot\|_\xi$ is the pull-back of the absolute value from some $K$-Banach algebra $\mathcal{H}$ that is a field under a homomorphism $\phi\colon A \to \mathcal{H}$. Since $\mathcal{H}$ is a field, the kernel of $\phi$ must be a prime ideal of $A$ and so defines an element of $\operatorname{Spec} A$ (or we just take $\mathcal{H}$ as a field, forgetting the absolute value).

It remains to show that the map $\operatorname{Berk} A \to \operatorname{Spec} A$ obtained in this way is continuous. Let $U_f = \{\xi \in \operatorname{Spec} A : f(\xi) \neq 0\}$ be a basic open set in $\operatorname{Spec} A$. Its pull-back to $\operatorname{Berk} A$ is $\{\xi \in \operatorname{Berk} A : \|f\|_\xi \neq 0\} = U_{f,0,\infty}$ and therefore open, too. ❑

The composition of the two maps $X(K) \to X^{\mathrm{an}} \to \operatorname{Spec} A$ is the inclusion of $X(K)$ (the set of maximal ideals of $A$) into $\operatorname{Spec} A$ (the set of prime ideals of $A$).

The map is actually surjective: let $\xi \in \operatorname{Spec} A$; this gives us a $K$-algebra homomorphism $A \to R$ into an integral domain finitely generated over $K$. One can show that one can always define an absolute value on such an $R$ that extends the absolute value on $K$. Pulling back to $A$, we obtain a multiplicative seminorm; this is a preimage of $\xi$.

It is also true that $X^{\mathrm{an}}$ is (even path-)connected when $X$ is connected as an algebraic variety and that $X^{\mathrm{an}}$ is locally compact. We will see this concretely for $X = \mathbb{A}^1$ in the next section.

## 5. The Berkovich affine and projective line

In the following, we will take a closer look at the Berkovich affine line over $\mathbb{C}_p$, $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} = \operatorname{Berk} \mathbb{C}_p[x]$. According to the classification of Definition 4.5, we have four types of points in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$. The type 1 points recover the points in $\mathbb{A}^1(\mathbb{C}_p) = \mathbb{C}_p$ as in Lemma 4.8. The type 2 and 3 points correspond to closed disks $D(a,r)$ with $r > 0$ in, respectively, not in, the value group $p^{\mathbb{Q}}$ of $\mathbb{C}_p$. We will reserve the notation $\zeta_{a,r}$ or $\zeta_D$ where $D = D(a,r)$ for these points. We will frequently identify $a$ with $\zeta_{a,0}$, however. Finally, the type 4 points correspond to nested sequences of disks with empty intersection; these points are somewhat annoying, but don't usually give us problems. They are necessary to make the space locally compact. Two nested sequences of disks $(D(a_n,r_n))_r$ and $(D(a'_n,r'_n))_n$ with empty intersection define the same type 4 point if and only if $D(a_n,r_n) \cap D(a'_n,r'_n) \neq \emptyset$ for all $n$.

Now fix $\xi \in \mathbb{C}_p$ and consider real numbers $0 \leq r_0 < r_1$. Then there is a map

$$[r_0, r_1] \longrightarrow \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}, \qquad r \longmapsto \|\cdot\|_{\xi,r} = \zeta_{\xi,r}\,.$$

This map is continuous, since for any $f \in \mathbb{C}_p[x]$, the map

$$r \longmapsto \|f\|_{\xi,r} = \max\{|a_j|_p r^j : 0 \leq j \leq n\}$$

is continuous, where $f(x+\xi) = a_0 + a_1 x + \ldots + a_n x^n$. The map is also clearly injective. Since $[r_0, r_1]$ is compact, the map is actually a homeomorphism onto its image. We write $[\zeta_{\xi,r_0}, \zeta_{\xi,r_1}]$ for this image.

Now consider two points $\xi, \eta \in \mathbb{C}_p$ and let $\delta = |\xi - \eta|_p$ their distance. Then $D(\xi,\delta) = D(\eta,\delta)$ ("every point in a disk is a center"). Define

$$\xi \vee \eta := \zeta_{\xi,\delta} = \zeta_{\eta,\delta}\,.$$

Then

$$\gamma_{\xi,\eta}\colon [0, 2\delta] \longrightarrow \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}, \qquad r \longmapsto \begin{cases} \zeta_{\xi,r} & \text{if } 0 \leq r \leq \delta, \\ \zeta_{\eta,2\delta - r} & \text{if } \delta \leq r \leq 2\delta \end{cases}$$

is a continuous path in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ joining $\xi$ and $\eta$, whose image is $[\xi, \xi \vee \eta] \cup [\eta, \xi \vee \eta]$.

We can extend the definition of $\xi \vee \eta$ to arbitrary points in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$. We first observe that

$$\|x - \eta\|_{\xi,r} = \begin{cases} \|x-\eta\|_{\eta,r} = r & \text{if } \eta \in D(\xi,r), \\ |\xi - \eta| > r & \text{if } \eta \notin D(\xi,r). \end{cases}$$

This shows that $D(\xi,r)$ is uniquely determined by $\|\cdot\|_{\xi,r}$ and also shows that $D_1 \subset D_2$ holds for two closed disks if and only if $\|\cdot\|_{D_1} \leq \|\cdot\|_{D_2}$ (the 'only if' part follows also from the characterization of $\|\cdot\|_{\xi,r}$ as the sup norm on $D(\xi,r)$). This prompts us to define

$$\xi \leq \xi' \iff \forall f \in \mathbb{C}_p[x]\colon \|f\|_\xi \leq \|f\|_{\xi'}$$

for arbitrary points $\xi, \xi' \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$. Furthermore, if $\xi \leq \xi'$, then we write $[\xi, \xi']$ for the set of points $\xi''$ such that $\xi \leq \xi'' \leq \xi'$. If $\xi$ and $\xi'$ are points of type 1, 2 or 3, then it is clear by the above that $[\xi, \xi']$ is homeomorphic to a closed interval in $\mathbb{R}$. Regarding type 4 points, we have the following.

5.1. **Lemma.** *Let $\xi \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ be a type 4 point, represented by the nested sequence of disks $D(a_n, r_n)$ with empty intersection. If $\xi' \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ with $\xi \le \xi'$, then either $\xi' = \xi$, or else $\xi'$ is a point of type 2 or 3, corresponding to a disk $D(a, r)$ such that $D(a_n, r_n) \subset D(a, r)$ for all sufficiently large $n$.* **LEMMA** comparison with type 4 points

*Let $r_\infty = \lim_{n\to\infty} r_n$. If $\xi'$ corresponds to $D(a, r)$, then*

$$\gamma_{\xi,\xi'} \colon [r_\infty, r] \longrightarrow \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}, \qquad \rho \longmapsto \begin{cases} \zeta_{a_n,\rho} & \text{if } \rho \ge r_n, \\ \xi & \text{if } \rho = r_\infty \end{cases}$$

*is a continuous map with image $[\xi, \xi']$.*

*Proof.* Let $\xi' \ge \xi$ and assume first that $\xi'$ is of type 2 or 3, so $\xi' = \zeta_{a,r}$ for some $a \in \mathbb{C}_p$ and $r > 0$. We claim that $D(a, r) \cap D(a_n, r_n) \ne \emptyset$ for all $n$. Otherwise, we would have empty intersection for all sufficiently large $n$, and then

$$\|x - a\|_{\xi'} = r < \lim_{n\to\infty} |a - a_n| = \lim_{n\to\infty} \|x - a\|_{a_n,r_n} = \|x - a\|_\xi ,$$

which contradicts $\xi \le \xi'$. If $n$ is large enough so that $r_n \le r$, then we must therefore have $D(a_n, r_n) \subset D(a, r)$.

Now assume that $\xi'$ is of type 1 or 4. If $\xi'$ is of type 1, then $\|x - \eta\|_{\xi'}$ is zero when $\eta = \xi'$, whereas $\|x-\eta\|_\xi \ge r_\infty > 0$ for all $\eta$, so this is not possible. If $\xi'$ is of type 4, say represented by the nested sequence of disks $D(a'_n, r'_n)$, then $\xi \le \xi' \le \zeta_{a'_n,r'_n}$ for all $n$. By the argument above, it follows that $D(a_n, r_n) \cap D(a'_m, r'_m) \ne \emptyset$ for all $m, n$. But this exactly means that $\xi = \xi'$.

It remains to show that $\gamma_{\xi,\xi'}$ is well-defined and continuous. For the first, note that $\zeta_{a_n,\rho} = \zeta_{a_{n+1},\rho}$ when $\rho \ge r_n$, since $D(a_n, \rho) = D(a_{n+1}, \rho)$. For the second, consider $f \in \mathbb{C}_p[x]$. We have to show that

$$[r_\infty, r] \ni \rho \longmapsto \|f\|_{\gamma_{\xi,\xi'}(\rho)}$$

is continuous. This is clear on $(r_\infty, r]$, and it follows for the left endpoint by the definition of $\|\cdot\|_\xi$, which implies that

$$\|f\|_\xi = \lim_{\rho\searrow r_\infty} \|f\|_{a_{n(\rho)},\rho}$$

(with $n(\rho)$ such that $\rho \ge r_n$). ❑

For $\xi' \le \xi$, we define the path $\gamma_{\xi,\xi'}$ as the reversal of the path $\gamma_{\xi',\xi}$.

5.2. **Theorem.** *Any two points $\xi, \xi' \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ have a least upper bound $\xi \vee \xi'$ (with respect to the ordering introduced above). The path $\gamma_{\xi,\xi\vee\xi'} + \gamma_{\xi\vee\xi',\xi'}$ connects the two points.* **THM** $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ is path-connected

*Proof.* This is clear if neither $\xi$ nor $\xi'$ are of type 4: if $\xi = \zeta_{a,r}$ and $\xi' = \zeta_{a',r'}$, then $\xi \vee \xi' = \zeta_{a,\rho} = \zeta_{a',\rho}$, where $\rho = \max\{r, r', |a - a'|\}$. If $\xi \le \xi'$ or $\xi' \le \xi$, then the statement is also clear (with $\xi \vee \xi' = \xi$ or $\xi'$). Otherwise, we can represent $\xi$ and $\xi'$ by nested sequences of disks $D(a_n, r_n)$ and $D(a'_n, r'_n)$, respectively, such that $D(a_n, r_n) \cap D(a'_n, r'_n) = \emptyset$ for $n$ large enough. For all such $n$, $\zeta_{a_n,r_n} \vee \zeta_{a'_n,r'_n}$ is the same and therefore agrees with $\xi \vee \xi'$.

The statement on path-connectedness is then clear. ❑

We define analogues of closed disks in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$.

5.3. **Definition.** Let $a \in \mathbb{C}_p$ and $r \geq 0$. We set

**DEF** closed disk in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$

$$\mathcal{D}(a,r) = \{\xi \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} : \xi \leq \zeta_{a,r}\} = \{\xi \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} : \|x-a\|_\xi \leq r\}.$$

and call this a *closed disk in* $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$. We also define the corresponding *open disk in* $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ to be

$$\mathcal{D}(a,r)^- = \{\xi \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} : \|x-a\|_\xi < r\} = U_{x-a,-\infty,r}. \qquad \diamondsuit$$

The equality in the definition of $\mathcal{D}(a,r)$ can be seen as follows. '$\subset$' is clear. To show '$\supset$', assume that $\|x-a\|_\xi \leq r$. Then for any $b \in \mathbb{C}_p$, we have

$$\|x-b\|_\xi = \|(x-a)+(a-b)\|_\xi \leq \max\{\|x-a\|_\xi, |a-b|\} \leq \max\{r, |a-b|\} = \|x-b\|_{a,r}.$$

Since every $f \in \mathbb{C}_p[x]$ is a constant times a product of such terms, it follows that $\xi \leq \zeta_{a,r}$.

5.4. **Lemma.**

**LEMMA** sub-basis of topology

(1) *Let* $f = (x-\alpha_1)\cdots(x-\alpha_n) \in \mathbb{C}_p[x]$ *be non-constant and let* $a \in \mathbb{R}$. *Then there are* $r_1, \ldots, r_n \geq 0$ *such that*

$$\{\xi \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} : \|f\|_\xi \leq a\} = \mathcal{D}(\alpha_1, r_1) \cup \ldots \cup \mathcal{D}(\alpha_n, r_n)$$

*and*

$$\{\xi \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} : \|f\|_\xi < a\} = \mathcal{D}(\alpha_1, r_1)^- \cup \ldots \cup \mathcal{D}(\alpha_n, r_n)^-$$

(2) *The open disks* $\mathcal{D}(a,r)^-$ *and the complements of closed disks* $\mathcal{D}(a,r)$ *generate the topology of* $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ *(i.e., every open set is a union of finite intersections of such sets).*

(3) *Two (open or closed) disks in* $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ *are either disjoint or one is contained in the other.*

*Proof.*

(1) Exercise.

(2) We have $U_{f,a,b} = U_{f,-\infty,b} \cap U_{f,a,\infty}$. By part (1), we can write $U_{f,-\infty,b}$ as a union of open disks, and we can write $U_{f,a,\infty}$, which is the complement of the set of $\xi$ such that $\|f\|_\xi \leq a$, as the complement of a finite union of closed disks, which is the same as a finite intersection of complements of closed disks. So each basic open set $U_{f,a,b}$ is a finite intersection of open disks and complements of closed disks; this implies the claim.

(3) Assume that $\mathcal{D}(a,r)$ and $\mathcal{D}(a',r')$ are not disjoint, so that there is $\xi$ with $\|x-a\|_\xi \leq r$ and $\|x-a'\|_\xi \leq r'$, w.l.o.g. such that $r \geq r'$. Then

$$|a-a'| \leq \max\{\|x-a\|_\xi, \|x-a'\|_\xi\} \leq r.$$

Now let $\xi' \in \mathcal{D}(a',r')$ be arbitrary. Then

$$\|x-a\|_{\xi'} \leq \max\{\|x-a'\|_{\xi'}, |a-a'|\} \leq \max\{r', r\} = r,$$

so $\xi' \in \mathcal{D}(a,r)$. Hence $\mathcal{D}(a',r') \subset \mathcal{D}(a,r)$. The case when one or both disks are open is similar. ❑

Together with (2) (and the fact that $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ is a union of open disks), statement (3) says that the open disks, from which a finite number of (pairwise disjoint) closed disks is removed, form a basis of the topology: every open set is a union of such sets.

5.5. **Lemma.** *Let $a \in \mathbb{C}_p$ and $r \geq 0$.*

**LEMMA** $\mathcal{D}(a,r)$ is compact

(1) *The set of type 1 points in $\mathcal{D}(a,r)$ is $D(a,r)$, and $D(a,r)$ is dense in $\mathcal{D}(a,r)$.*

(2) *$\mathcal{D}(a,r)$ is compact.*

*Proof.* The first claim in (1) is clear (and holds in a similar way for open disks). For the second claim, consider the intersection of a basic open set as above with $\mathcal{D}(a,r)$. The set of type 1 points contained in this intersection is the intersection of $D(a,r)$ with the open disk in $\mathbb{C}_p$ corresponding to the open disk, minus the union of the closed disks in $\mathbb{C}_p$ corresponding to the closed disks that were removed. Any such set is non-empty (unless the whole disk $D(a,r)$ is removed, but then the basic open set has empty intersection with $\mathcal{D}(a,r)$).

For (2), we can assume without loss of generality that $a = 0$. We know that $\xi \in \mathcal{D}(0,r) \iff \|\cdot\|_\xi \leq |\cdot|_r$, so the image of $\mathcal{D}(0,r)$ in $\mathbb{R}^{\mathbb{C}_p[x]}$ under the map

$$\Phi\colon \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} \longrightarrow \mathbb{R}^{\mathbb{C}_p[x]}, \qquad \xi \longmapsto (\|f\|_\xi)_{f \in \mathbb{C}_p[x]}$$

is the intersection of $\mathrm{im}(\Phi)$ with the product $C = \prod_f [0, |f|_r]$. The definition of the topology on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ is equivalent to saying that it is the subspace topology induced by the map $\Phi$ above (with the product topology on $\mathbb{R}^{\mathbb{C}_p[x]}$). The product $C$ of compact intervals is itself compact by Tychonoff's Theorem. On the other hand, the conditions defining a multiplicative seminorm are closed conditions, so the image of $\Phi$ is closed. Now $\Phi(\mathcal{D}(0,r))$ is the intersection of a closed set and a compact set, so it is itself compact. Since $\Phi$ is a homeomorphism onto its image, it follows that $\mathcal{D}(0,r)$ is compact as well. ❑

5.6. **Corollary.** *The space $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ is locally compact.*

**COR** $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ is locally compact

*Proof.* We must show that for every point $\xi \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ and every open subset $U$ of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ containing $\xi$, there is a compact neighborhood $V$ of $\xi$ contained in $U$. Since $U$ is open, $U$ contains an open set of the form $\mathcal{D}(a,r)^- \setminus \bigcup_j \mathcal{D}(a_j, r_j)$ that in turn contains $\xi$. This means that $\|x-a\|_\xi < r$ and $\|x-a_j\|_\xi > r_j$. For $r' < r$ sufficiently close to $r$ and for $r'_j > r_j$ sufficiently close to $r_j$, we still have $\|x-a\|_\xi < r'$ and $\|x-a_j\|_\xi > r'_j$. Then

$$\xi \in V := \mathcal{D}(a,r') \setminus \bigcup_j \mathcal{D}(a_j, r'_j)^- \subset U\,.$$

Now $V$ is compact (we intersect the compact set $\mathcal{D}(a,r')$ with a closed set) and contains the open neighborhood $\mathcal{D}(a,r')^- \setminus \bigcup_j \mathcal{D}(a_j, r'_j)$ of $\xi$. ❑

We now look a bit closer at the tree structure of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$.

5.7. **Definition.** The *diameter* of a point $\xi \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ is

**DEF** diameter

$$\mathrm{diam}(\xi) := \inf\{\|x-a\|_\xi : a \in \mathbb{C}_p\}\,. \qquad \diamondsuit$$

Then $\mathrm{diam}(\zeta_{a,r}) = r$, and for a type 4 point $\xi$ represented by a nested sequence of disks $D(a_n, r_n)$, we have $\mathrm{diam}(\xi) = \lim_{n\to\infty} r_n > 0$.

We use this notion to define two metrics.

5.8. **Definition.** For $\xi, \xi' \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$, we define the *small metric* by **DEF** small and big metric

$$\begin{aligned} d(\xi, \xi') &= 2\operatorname{diam}(\xi \vee \xi') - \operatorname{diam}(\xi) - \operatorname{diam}(\xi') \\ &= (\operatorname{diam}(\xi \vee \xi') - \operatorname{diam}(\xi)) + (\operatorname{diam}(\xi \vee \xi') - \operatorname{diam}(\xi')) \end{aligned}$$

and in the case that $\xi, \xi'$ are both not of type 1, the *big metric* by

$$\rho(\xi, \xi') = c(2\log\operatorname{diam}(\xi \vee \xi') - \log\operatorname{diam}(\xi) - \log\operatorname{diam}(\xi')) .$$ ♢

So the small metric gives the total change of diameter as we move along the path joining $\xi$ to $\xi'$, whereas the big metric does the same for the change in (additive) valuation ($= -c\log\operatorname{diam}$). Both are indeed metrics, $d$ on all of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ and $\rho$ on the 'hyperbolic part' of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$, which consists of the points of types 2, 3 and 4. The former has the advantage that it is also defined on the points of type 1, but the latter is in some sense more natural. It can be seen as analogous to the hyperbolic metric on the upper half plane in $\mathbb{C}$ (which is given by $ds^2 = (dx^2 + dy^2)/y^2$ when $x$ and $y$ are the real and imaginary parts), which is invariant under the action of $\mathrm{PSL}(2, \mathbb{R})$ by Möbius transformations. This analogous property will turn out to be satisfied (with respect to $\mathrm{PGL}(2, \mathbb{C}_p)$) by the big metric.

Now we introduce the kind of tree structure that shows up here. (For more information, consult [BR, Appendix B].)

5.9. **Definition.** An $\mathbb{R}$-*tree* is a metric space $(T, d)$ such that for any two points $x, y \in T$, there is a unique path $[x, y]$ in $T$ joining $x$ to $y$, which is a geodesic segment (this means that the map $\gamma\colon [a, b] \to T$ giving the path can be chosen so that $d(\gamma(u), \gamma(v)) = |u - v|$ for all $u, v \in [a, b]$). **DEF** $\mathbb{R}$-tree

A point $x \in T$ is a *branch point* if $T \setminus \{x\}$ has at least three connected components (in the metric topology). $x$ is an *endpoint* if $T \setminus \{x\}$ is connected. If $T \setminus \{x\}$ has exactly two connected components, then $x$ is said to be *ordinary*. The $\mathbb{R}$-tree $T$ is said to be *finite*, if it is compact and has only finitely many branch points and endpoints.

The *strong topology* on the $\mathbb{R}$-tree is the metric topology on $T$. To define the *weak topology*, we define a *tangent direction* at $x \in T$ to be an equivalence class of paths $[x, y]$ with $y \neq x$, where two paths are equivalent when they share an initial segment. The tangent directions at $x$ are in one-to-one correspondence with the connected components of $T \setminus \{x\}$. For a tangent direction $v$ at $x$, we set $B_{x,v} = \{y \in T : y \neq x, [x, y] \in v\}$ (which is the connected component of $T \setminus \{x\}$ corresponding to $v$). Then the weak topology on $T$ is the topology generated by the sets $B_{x,v}$. ♢

The set $B_{x,v}$ can be interpreted as the set of points of $T$ that can be 'seen' from $x$ when looking into direction $v$. This is why the weak topology is also called the 'observer's topology'.

Now the following is a fact.

5.10. **Theorem.** **THM** $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ as $\mathbb{R}$-tree

(1) *The small metric $d$ turns $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ into an $\mathbb{R}$-tree. The Berkovich topology on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ is the weak topology on this $\mathbb{R}$-tree.*

(2) *The big metric $\rho$ turns $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus \mathbb{C}_p$ into an $\mathbb{R}$-tree. The subspace topology on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus \mathbb{C}_p$ is again the weak topology on this $\mathbb{R}$-tree.*

*Proof.* See [BR, Section 1.4], where the analogous statements are shown for $\mathcal{D}(0,1)$. For $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ the argument is the same (except that there is no natural way to pick a root). ❑

Compactifying the $\mathbb{R}$-tree by adding a point at infinity (see below) we obtain the *universal dendrite* that was first constructed by Ważewski in his thesis in 1923. Recent work by Hrushovski, Loeser and Poonen[1] shows that $V^{\mathrm{an}}$ can be embedded in $\mathbb{R}^{2d+1}$ for any quasi-projective $d$-dimensional $\mathbb{C}_p$-variety $V$ (and $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ can be embedded in $\mathbb{R}^2$).

We can classify the points in the $\mathbb{R}$-tree structure.

(1) Points of types 1 and 4 are endpoints.

(2) Points of type 2 are branch points, and the set of tangent directions corresponds (canonically up to an automorphism of $\mathbb{P}^1_{\bar{\mathbb{F}}_p}$) to $\mathbb{P}^1(\bar{\mathbb{F}}_p)$.

(3) Points of type 3 are ordinary.

Let $\xi \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ be any point. If $\xi'$ is another point, then there are three possibilities: $\xi > \xi'$, $\xi < \xi'$, or neither. In the last two cases, the path from $\xi$ to $\xi'$ starts 'going up' to $\xi \vee \xi'$ ($= \xi'$ in the second case). So the points $\xi' \neq \xi$ such that $\xi \not> \xi'$ constitute the set $B_{\xi,\mathrm{up}}$. If $\xi$ is of type 1 or 4, then there are no $\xi'$ with $\xi > \xi'$, so 'up' is the only direction.

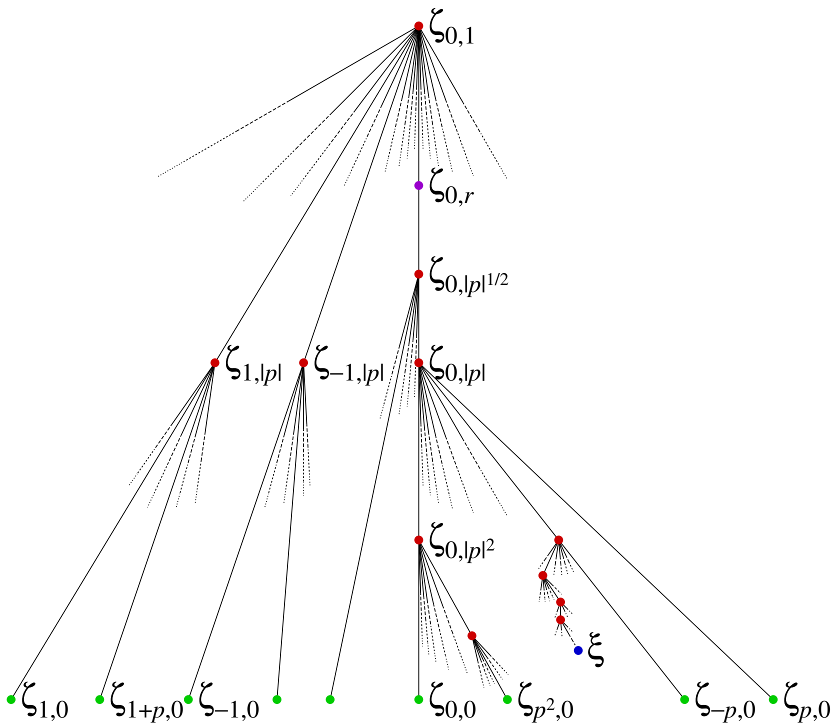
Now consider $\xi = \zeta_{a,r}$ of type 3. If $\xi' < \xi$, then there is some $\xi' \le \zeta_{a',r'} < \xi$, and we have $D(a',r') \subset D(a,r)$, so $|a'-a|_p \le r$. Since $r \notin |\mathbb{C}_p^\times|_p$, we can choose $r'$ such that $|a'-a|_p < r' < r$, but then $\zeta_{a',r'} = \zeta_{a,r'}$, so the path from $\xi$ to $\xi'$ shares an initial segment with $[\zeta_{a,r}, \zeta_{a,0}]$. This shows that other than 'up', there is exactly one tangent direction 'down'.

Finally, when $\xi = \zeta_{a,r}$ is of type 2, then by a similar argument, for any $\xi' < \xi$ there is $a' \in D(a,r)$ such that $\xi' \in \mathcal{D}(a',r)^-$. If two points $\xi', \xi''$ are in the same open disk, then the paths from $\xi$ to these two points share an initial segment; otherwise $\xi' \vee \xi'' = \xi$ and they do not. So the directions other than 'up' at $\xi$ are in one-to-one correspondence with open disks of radius $r$ contained in $\mathcal{D}(a,r)$. To show that the set of such disks corresponds to $\bar{\mathbb{F}}_p$, we assume that $a = 0$ and $r = 1$ (we can shift and scale to reduce to this case). Two disks $\mathcal{D}(a',1)^-$ and $\mathcal{D}(a'',1)^-$ contained in $\mathcal{D}(0,1)$ are equal if and only if $|a'-a''|_p < 1$, which means that $a', a'' \in R$ (the valuation ring of $\mathbb{C}_p$) have the same image in the residue class field $\bar{\mathbb{F}}_p$. So we see that the 'downward' directions at a type 2 point correspond to the elements of $\bar{\mathbb{F}}_p$; together with the 'up' direction, which we can let correspond to $\infty$, we get $\mathbb{P}^1(\bar{\mathbb{F}}_p)$. (For $\zeta_{0,1}$ the correspondence is canonical. For other points it is so only up to an automorphism of $\mathbb{P}^1_{\bar{\mathbb{F}}_p}$, since there is a choice involved in the shifting and scaling.)

If $\xi = \zeta_{a,r}$ is a point of type 2 or 3, then the basic open set $B_{\xi,v}$ of the weak topology is the complement of $\mathcal{D}(a,r)$ when $v =$ 'up' and is $\mathcal{D}(a',r)^-$ with $a'$ as above when $v$ is a 'downward' direction. For $\xi$ of type 1 or 4, $B_{\xi,v}$ is just $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus \{\xi\}$. This shows that the weak topology of the $\mathbb{R}$-tree agrees with the Berkovich topology.

Here is a rough sketch of $\mathcal{D}(0,1)$:

[1] E. Hrushovski, F. Loeser and B. Poonen, *Berkovich spaces embed in Euclidean spaces,* L'Enseignement Math. **60** (2014), no. 3–4, 273–292.




Points of type 1, 2, 3 and 4 are green, red, purple and blue, respectively.

Let $T$ be an $\mathbb{R}$-tree and let $S \subset T$ be nonempty and finite. Then one can consider the *convex hull* $\Gamma$ of $S$; this is the union of the paths joining the points in $S$. Then $\Gamma$ is a finite $\mathbb{R}$-tree and there is a natural deformation retraction $\tau_\Gamma \colon T \to \Gamma$ that sends any point $x \in T$ to the point of $\Gamma$ that is hit first by the unique path from $x$ to any fixed point in $\Gamma$. It can be shown that the weak topology on $T$ is the weakest topology that makes all the maps $\tau_\Gamma$ continuous (where we take, say, the metric topology on $\Gamma$). (This is an exercise.)

Now we want to introduce the Berkovich projective line, $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$. There are several ways of constructing it.

(1) We set $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} = \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} \cup \{\infty\}$, where $\infty$ is a point of type 1, whose open neighborhoods are the complements of compact subsets of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ together with the point $\infty$. (This is the *one-point compactification* from general topology.)

(2) We observe that the map $\mathbb{C}_p \to \mathbb{C}_p$, $z \mapsto z^{-1}$ induces a continuous involution $\phi$ on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus \{0\}$ (that is given on points not of type 4 by $\zeta_{a,r} \mapsto \zeta_{a^{-1},r/|a|^2}$ when $|a| > r$ and $\zeta_{0,r} \mapsto \zeta_{0,1/r}$ for $r > 0$). One can then 'glue' two copies of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ along $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus \{0\}$ via this map. This identifies a neighborhood of $\infty$ with a neighborhood of 0 (and so shows in particular that $\infty$ is in no way special). One can also show that the big metric is invariant under $\phi$ and also under affine maps $z \mapsto az + b$ (Exercise).

(3) One obtains the same result by gluing two copies of $\mathcal{D}(0,1)$ along the 'spheres' $\mathcal{D}(0,1) \setminus \mathcal{D}(0,1)^-$ via $\phi$. Since $\mathcal{D}(0,1)$ is compact, this shows that $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ is compact.

(4) Similar to the Proj construction in algebraic geometry, there is a 'Berkovich Proj' taking a graded finitely generated algebra $A$ over a complete non-archimedean field as input and having a Berkovich type space as output, whose points correspond to equivalence classes of multiplicative seminorms on $A$ whose kernel doe

not contain the irrelevant ideal. Applying this to the polynomial ring $\mathbb{C}_p[x, y]$ with the standard grading, this results in $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$. We will discuss this in more detail below. An advantage of this approach is that it makes it easy to see that a morphism $\mathbb{P}^1_{\mathbb{C}_p} \to \mathbb{P}^1_{\mathbb{C}_p}$ induces a continuous map $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} \to \mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$.

Recall that a *graded ring* is a ring $R$ coming with a direct sum decomposition $R = \bigoplus_{n\ge 0} R_n$ as an additive group such that for all $f \in R_m$ and $g \in R_n$ we have $fg \in R_{m+n}$. The elements of $R_n$ are said to be *homogeneous of degree $n$*. An ideal $I$ of $R$ is said to be *homogeneous* if it is generated by homogenous elements. If $K$ is a field and the decomposition is a direct sum of $K$-vector spaces, then $R$ is a graded $K$-algebra. In algebraic geometry, $\operatorname{Proj} R$ is the set of all homogeneous prime ideals of $R$ that do not contain the *irrelevant ideal* $\bigoplus_{n\ge 1} R_n$, with the Zariski topology. For example, $\mathbb{P}^n_K = \operatorname{Proj} K[X_0, X_1, \ldots, X_n]$ with the usual grading of the polynomial ring. The 'Berkovich Proj' construction does something similar for Berkovich spaces.

5.11. **Definition.** Let $A$ be a finitely generated graded $K$-algebra with $A_0 = K$, where $K$ is a complete and algebraically closed non-archimedean field. We consider the set of multiplicative seminorms on $A$ whose kernel does not contain the irrelevant ideal. We declare that two seminorms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if there is $C > 0$ such that $\|F\|' = C^d \|F\|$ for all $F \in A_d$ (i.e., $F \in A$ homogeneous of degree $d$). Then the *projective Berkovich space* associated to $A$, $\operatorname{PBerk} A$, is the set of equivalence classes of these seminorms. **DEF** Berkovich Proj

Let $a_1, \ldots, a_m$ be homogeneous generators of the irrelevant ideal of $A$. Then in each equivalence class, there are *normalized* seminorms $\|\cdot\|$ with the property that $\max_j \|a_j\| = 1$, and all equivalent normalized seminorms agree on all homogeneous elements. We define the topology on $\operatorname{PBerk} A$ to be the weakest one such that $[\|\cdot\|] \mapsto \|F\|$ is continuous for every homogeneous $F \in A$, where $\|\cdot\|$ is a normalized representative of the class $[\|\cdot\|]$.

If $V = \operatorname{Proj} A$ is the projective $K$-variety defined by $A$, then we also write $V^{\mathrm{an}}$ for $\operatorname{PBerk} A$. ♢

The notation $\operatorname{PBerk} A$ is not standard.

In analogy with the usual construction in algebraic geometry, we can then define the *Berkovich projective line* to be **DEF** $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$

$$\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} = \operatorname{PBerk} \mathbb{C}_p[X, Y]\,,$$

where the polynomial ring $\mathbb{C}_p[X, Y]$ has its usual grading.

We find the usual two embeddings of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ into $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ by pulling back seminorms under the two maps $\mathbb{C}_p[X, Y] \to \mathbb{C}_p[x]$, $F(X, Y) \mapsto F(x, 1)$ and $F(X, Y) \mapsto F(1, x)$. The equivalence class we do not obtain under the first map satisfies $\|Y\| = 0$; it corresponds to the type 1 point 0 under the second map and represents the point at infinity.

It is then natural to consider sets of the form $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus \mathcal{D}(a, r)^-$ as closed and sets of the form $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus \mathcal{D}(a, r)$ as open Berkovich disks ('around infinity') in $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$. Then the open Berkovich disks generate the topology of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$, and a basis of the topology is given by the open disks (including the whole space) minus finitely many closed disks.

Since $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ without the points of type 1 is the same as $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ without the points of type 1, which is an $\mathbb{R}$-tree with respect to the big metric, it follows by the results mentioned in (2) above that $\mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}_p}) = \mathrm{PGL}(2, \mathbb{C}_p)$ acts on this $\mathbb{R}$-tree by isometries. This action is transitive on the set of type 2 points, and the stabilizer of the point $\zeta_{0,1}$ is $\mathrm{PGL}(2, R)$ (where $R$ is the valuation ring of $\mathbb{C}_p$), with the tangent directions at $\zeta_{0,1}$ being permuted transitively (by $\mathrm{PGL}(2, \bar{\mathbb{F}}_p) = \mathrm{Aut}(\mathbb{P}^1_{\bar{\mathbb{F}}_p})$ via the canonical map $\mathrm{PGL}(2, R) \to \mathrm{PGL}(2, \bar{\mathbb{F}}_p)$).

A type 2 point $\zeta$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ can then be interpreted as corresponding to a reduction map $\mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\bar{\mathbb{F}}_p)$, up to an automorphism of the target. The map is given by associating to $z \in \mathbb{P}^1(\mathbb{C}_p)$ the tangent direction at $\zeta$ in which the type 1 point corresponding to $z$ can be seen. For $\zeta = \zeta_{0,1}$ we get the usual reduction map. Such a reduction map comes from a 'model' of $\mathbb{P}^1$ over $R$ of the form $\mathbb{P}^1_R$; the reduction map depends on how we identify the generic fiber of $\mathbb{P}^1_R$ with $\mathbb{P}^1_{\mathbb{C}_p}$. More generally, we call any finite $\mathbb{R}$-subtree of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{C}_p)$ with the big metric that is the convex hull of finitely many points of type 2 a *skeleton* of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$. Any skeleton corresponds to a model of $\mathbb{P}^1_{\mathbb{C}_p}$ over $R$ that is usually more complicated than $\mathbb{P}^1_R$ in that its special fiber (i.e., the base change to $\bar{\mathbb{F}}_p$) is a configuration of several $\mathbb{P}^1$'s arranged in tree form. (Possibly we return to that later.) **DEF** Skeleton

If we modify the definition of the diameter by defining it on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ as

$$\mathrm{diam}'(\xi) = \begin{cases} \mathrm{diam}(\xi) & \text{if } \xi \in \mathcal{D}(0,1) \\ \mathrm{diam}(\phi(\xi)) & \text{if } \xi \in \mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus \mathcal{D}(0,1), \end{cases}$$

where $\phi$ is the involution induced by $z \mapsto 1/z$, and set $\xi \vee' \eta$ to be the point where $[\xi, \zeta_{0,1}]$ and $[\eta, \zeta_{0,1}]$ first meet, then we can define a small metric $d'$ on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ in the same way as before. Then $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ can be identified with the $\mathbb{R}$-tree given by the small metric. However, the small metric is not invariant under automorphisms in general (the point $\zeta_{0,1}$ plays a special role in its definition; any automorphism that moves $\zeta_{0,1}$ will change the metric).

To conclude this section, we give another interpretation of the seminorm associated to a point in $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ that (contrary to the interpretation as the supremum norm on the associated disk when applied to polynomials) also works for rational functions. Consider first a point $\xi = \zeta_{a,r}$ of type 2. If $f$ is a nonzero polynomial with roots $\alpha_1, \ldots, \alpha_m$ in $D(a,r)$, then for all $\alpha$ in the closed set $D(a,r) \setminus \bigcup_{j=1}^m D(\alpha_j, r)^-$ we have $|f(\alpha)|_p = \|f\|_\xi$. To see this, we can assume that $a = 0$ and $f$ is monic. Write

$$f = (x - \alpha_1) \cdots (x - \alpha_m)(x - \alpha_{m+1}) \cdots (x - \alpha_n)\,,$$

where $\alpha_{m+1}, \ldots, \alpha_n$ are the roots of $f$ outside of $D(0,r)$. Then

$$\|f\|_\xi = \prod_{j=1}^n \|x - \alpha_j\|_\xi = \prod_{j=1}^n |\alpha - \alpha_j|_p = |f(\alpha)|_p\,,$$

since for $j \le m$, we have $\|x-\alpha_j\|_\xi = r = |\alpha-\alpha_j|_p$ (here we use that $\alpha \notin D(\alpha_j, r)^-$), and for $j > m$, we have $\|x - \alpha_j\|_\xi = |\alpha_j|_p = |\alpha - \alpha_j|_p$. So $|f|_p$ is constant and equal to $\|f\|_\xi$ on $D(a,r)$ outside a finite union of open disks contained in $D(a,r)$. We can apply this separately to the numerator and the denominator of a rational function $f = g/h \in \mathbb{C}_p(x)$, which gives the same statement for such an $f$. So for a type 2 point $\xi = \zeta_{a,r}$, the induced seminorm $\|\cdot\|_\xi$ on $\mathbb{C}_p(x)$ is the absolute value taken by any given function $f \in \mathbb{C}_p(x)$ on $D(a,r)$ outside finitely many smaller open disks (the set of disks to be removed depends on $f$); this is also the same as the absolute value taken on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus D(a,r)^-$ outside a finite union of smaller open disks (one of which is the 'exterior' $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus D(a,r)$). Note that when considering rational functions $f$ as quotients of two homogeneous polynomials in $\mathbb{C}_p[X,Y]$ of the same degree, then $\|f\|_\xi$ is independent of the choice of representative of the equivalence class of seminorms corresponding to $\xi$ (the scaling factor $C^d$ cancels).

For a point $\xi = a \in \mathbb{C}_p$ of type 1, we have of course $\|f\|_\xi = |f(a)|_p$ unless $a$ is a pole of $f$ (this also makes sense when $a = \infty$, with $f(\infty)$ interpreted as the value at zero of $f(1/x)$).

If $\xi = \zeta_{a,r}$ is of type 3, then we have to use a 'limit version' of the above. Since $r \notin |\mathbb{C}_p^\times|$, no two nonzero terms of the form $r^j|a_j|_p$ with $a_j \in \mathbb{C}_p$ can be equal and so we have $\|f\|_\xi = \max_j r^j|a_j|_p$ when $f(x+a) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is a nonzero polynomial. If $s \in |\mathbb{C}_p^\times|$ is sufficiently close to $r$, then all terms $s^j|a_j|_p$ will still be pairwise distinct, hence for $\alpha \in C_p$ such that $|\alpha - a|_p = s$ we will have $|f(\alpha)|_p = \max_j s^j|a_j|_p$, which is close to $\|f\|_\xi$. So for any given $\varepsilon > 0$ there is a closed annulus $A = D(a, r+\delta) \setminus D(a, r-\delta)^-$ such that $\big||f(\alpha)|_p - \|f\|_\xi\big| \le \varepsilon$ for all $\alpha \in A$.

The general statement is as follows (see Section 2.4 in [BR]).

5.12. **Lemma.** *Let $\xi \in \mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ and denote the corresponding multiplicative seminorm on $\mathbb{C}_p(x)$ as usual by $\|\cdot\|_\xi$. Then for every $f \in \mathbb{C}_p(x)$ (that does not have a pole at $\xi$ when $\xi$ is of type 1) we have that $\|f\|_\xi$ is the unique $\nu \in \mathbb{R}_{\ge 0}$ such that for every $\varepsilon > 0$ there is a closed neighborhood $U$ of $\xi$ in $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ such that $\big||f(\alpha)|_p - \nu\big| \le \varepsilon$ for all $\alpha \in \mathbb{P}^1(\mathbb{C}_p) \cap U$. The set $U$ can be taken to be a closed Berkovich disk minus a finite union of open Berkovich disks.*

**LEMMA** characterization of $\|\cdot\|_\xi$

*Proof.* Fix $f$. Since $\eta \mapsto \|f\|_\eta$ is continuous by definition of the Berkovich topology, the set $U = \{\eta : |\|f\|_\eta - \|f\|_\xi| \le \varepsilon\}$ is a closed neighborhood of $\xi$. It contains an open neighborhood of $\xi$ (for example, the set obtained by replacing '$\le \varepsilon$' by '$< \varepsilon$'), which contains an open neighborhood $V$ that is an open Berkovich disk minus finitely many closed Berkovich disks. Then $U$ contains the closure of $V$, which is the corresponding closed Berkovich disk minus the corresponding open Berkovich disks. We can always make $U$ smaller, so we can assume that $U$ has this form. Then $\big||f(\alpha)|_p - \|f\|_\xi\big| \le \varepsilon$ for $\alpha \in \mathbb{P}^1(\mathbb{C}_p) \cap U$ follows from the definition of $U$. Since $\mathbb{P}^1(\mathbb{C}_p) \cap U$ is dense in $U$, this property determines $\|f\|_\xi$ to within $\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\|f\|_\xi$ is characterized by it. ❑

In a similar way as for Berkovich spaces associated to affine $\mathbb{C}_p$-varieties, morphisms between projective $\mathbb{C}_p$-varieties induce continuous maps between the associated Berkovich spaces. For example, any rational function $\varphi \in \mathbb{C}_p(x)$ can be considered as a morphism $\mathbb{P}^1_{\mathbb{C}_p} \to \mathbb{P}^1_{\mathbb{C}_p}$ and therefore induces a map $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} \to \mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$. If $\varphi$ is not constant (otherwise it is quite clear what happens), then it follows from the discussion above that $\varphi(\xi) = \eta$ for type 2 points $\xi = \zeta_{a,r}$ and $\eta = \zeta_{a',r'}$ if and only if $\varphi$ maps a suitable set $D(a,r) \setminus \bigcup_j D(a_j, r)^-$ to a set of the form $D(a', r') \setminus \bigcup_j D(a'_j, r')$. This can be helpful when one is trying to figure out what $\varphi$ is doing on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$.

# 6. Analytic spaces and functions

In this section, we follow roughly part of Brian Conrad's text [AWS, Chapter 1].

Recall that a $K$-Banach algebra (for a complete field $K$ with absolute value) is a $K$-algebra $A$ with a (submultiplicative) norm $|\cdot|_A$ that restricts to the absolute value on $K$ and such that $A$ is complete with respect to this norm. So we have

$$|a|_A \neq 0 \quad \text{for } a \neq 0, \qquad |a+b|_A \le |a|_A + |b|_A \quad \text{and} \quad |ab|_A \le |a|_A \cdot |b|_A\,.$$

If $K$ is non-archimedean, then $|\cdot|_A$ also satisfies the ultrametric triangle inequality.

6.1. **Definition.** Let $A$ be a $K$-Banach algebra; we write the norm on $A$ as $|\cdot|_A$. A multiplicative seminorm $\|\cdot\|$ on $A$ is said to be *bounded*, if $\|f\| \le |f|_A$ for all $f \in A$. ♢

**DEF** bounded multiplicative seminorm

We note that it is sufficient to require $\|f\| \le C|f|_A$ for some constant $C > 0$, since this implies

$$\|f\| = (\|f^n\|)^{1/n} \le (C|f^n|_A)^{1/n} \le C^{1/n}|f|_A\,;$$

we obtain $\|f\| \le |f|_A$ by letting $n$ tend to infinity. So the notion of 'bounded multiplicative seminorm' does not change when we replace the norm on $A$ by an equivalent one (i.e., one that is bounded below and above by some positive multiple of $|\cdot|_A$).

6.2. **Definition.** Let $A$ be a $K$-Banach algebra. The *Berkovich space* $\operatorname{Berk} A$ associated to $A$ is the set of all bounded multiplicative seminorms on $A$, with the weakest topology that makes the maps $\|\cdot\| \mapsto \|f\|$ continuous for all $f \in A$. ♢

**DEF** Berkovich space associated to $A$

In a similar way as we did earlier, one shows the following.

6.3. **Theorem.** *Let $A$ be a $K$-Banach algebra. Then* $\operatorname{Berk} A$ *is a compact Hausdorff space, which is empty if and only if $A = \{0\}$.*

**THM** $\operatorname{Berk} A$ is compact and Hausdorff

We note that one can define $\operatorname{Berk} A$ for any (commutative) ring $A$ that is complete with respect to some absolute value (a *Banach ring*); it does not have to be an algebra over some complete field. One could take $A = \mathbb{Z}$ with the usual absolute value, for example (completeness follows trivially, since every Cauchy sequence must be eventually constant), or any ring with the trivial absolute value. It is an instructive exercise to work out the structure of $\operatorname{Berk} \mathbb{Z}$!

From now on, $K$ will be, as usual, a complete and algebraically closed non-archimedean field, for example, $K = \mathbb{C}_p$. There is an important class of $K$-Banach algebras.

6.4. **Definition.** Let $n \in \mathbb{Z}_{>0}$ and $r_1, \ldots, r_n > 0$. We define

**DEF** Tate algebra

$$K\langle r_1^{-1}x_1, \ldots, r_n^{-1}x_n\rangle := \Big\{ \sum_{i \in \mathbb{Z}^n_{\ge 0}} a_i x^i : r^i|a_i| \to 0 \text{ as } |i| \to \infty \Big\} \subset K[\![x_1, \ldots, x_n]\!]\,.$$

Here we use multi-index notation: for $i = (i_1, \ldots, i_n)$ we set $x^i = x_1^{i_1} \cdots x_n^{i_n}$ and $r^i = r_1^{i_1} \cdots r_n^{i_n}$; also $|i| = i_1 + \ldots + i_n$. Then

$$\Big|\sum_i a_i x^i\Big| := \max_i r^i|a_i|$$

defines a multiplicative norm on $A = K\langle r_1^{-1}x_1, \dots, r_n^{-1}x_n\rangle$ turning $A$ into a $K$-Banach algebra. We call $A$ a *Tate algebra* over $K$ when all $r_j \in |K^\times|$, otherwise $A$ is a *generalized Tate algebra* over $K$.

More generally, if $A$ is any $K$-Banach algebra, then we can define *(generalized) relative Tate algebras* $A\langle r_1^{-1}x_1, \dots, r_n^{-1}x_n\rangle$ in the same way. ♢

Note that by scaling the variables suitably, any Tate algebra in $n$ variables is isomorphic to the standard Tate algebra

$$K\langle x_1, \dots, x_n\rangle := K\langle 1^{-1}x_1, \dots, 1^{-1}x_n\rangle\,.$$

This is not true for generalized Tate algebras.

6.5. **Example.** We have $\operatorname{Berk}\mathbb{C}_p\langle x\rangle \cong \mathcal{D}(0,1)$. The map is induced by the inclusion $\mathbb{C}_p[x] \hookrightarrow \mathbb{C}_p\langle x\rangle$; the image consists of all seminorms on $\mathbb{C}_p[x]$ that are bounded by the restriction of the norm on the Tate algebra to the polynomial ring, which is exactly $|\cdot|_1 = \|\cdot\|_{0,1}$. By definition, this set is $\mathcal{D}(0,1)$. It also follows easily from the definitions that the map induces a homeomorphism. **EXAMPLE** $\mathcal{D}(0,1)$

More generally, the same argument shows that $\operatorname{Berk}\mathbb{C}_p\langle r^{-1}x\rangle \cong \mathcal{D}(0,r)$, for every $r > 0$. We see that there is some kind of correspondence between type 2 points and Tate algebras, and type 3 points and generalized Tate algebras. Quite explicitly, we obtain $\mathbb{C}_p\langle r^{-1}x\rangle$ as the completion of $\mathbb{C}_p[x]$ with respect to the norm corresponding to $\zeta_{0,r}$. ♣

We state the following facts without proof.

6.6. **Lemma.** *The Tate algebras over $K$ are noetherian (every ideal is finitely generated) unique factorization domains. They are also Jacobson rings (every prime ideal is the intersection of the maximal ideals it is contained in). Every ideal is closed.* **LEMMA** Properties of Tate algebras

*The norm is intrinsically defined as $|f| = \max_\phi |\phi(f)|$, where $\phi$ runs over all continuous $K$-algebra homomorphisms to $K$.*

*The Tate algebra $T = K\langle r_1^{-1}x_1, \dots, r_n^{-1}x_n\rangle$ has the following universal property. Given any $K$-Banach algebra $A$ and elements $a_1, \dots, a_n \in A$ with $|a_j|_A \le r_j$, there is a unique continuous $K$-algebra homomorphism $\phi\colon T \to A$ with $\phi(x_j) = a_j$.*

Let $A$ be a Tate algebra and $I \subset A$ an ideal. Then by the above, $I$ is closed. Define a real-valued function on $A/I$ by

$$|a + I|_{A/I} = \inf\{|a + b|_A : b \in I\}\,.$$

6.7. **Lemma.** *$|\cdot|_{A/I}$ is a norm on $A/I$ turning $A/I$ into a Banach algebra such that the canonical homomorphism $A \to A/I$ is continuous.* **LEMMA** induced norm

*Proof.* We have to show a number of things.

(1) $|a + I|_{A/I} = 0$ only for $a \in I$.
 If $|a + I|_{A/I} = 0$, then there is a sequence $(b_n)$ in $I$ such that $|a + b_n|_A \to 0$, hence $-b_n \to a$. Since $I$ is closed, this implies $a \in I$.

(2) $|(a + a') + I|_{A/I} \le \max\{|a + I|_{A/I}, |a' + I|_{A/I}\}$.
 This follows easily from the definition.

(3) $|aa' + I|_{A/I} \le |a + I|_{A/I} \cdot |a' + I|_{A/I}$.
For $b, b' \in I$ we have $|aa' + I|_{A/I} \le |(a + b)(a' + b')|_A \le |a + b|_A \cdot |a' + b'|_A$; taking the infimum over all $b$ and $b'$ yields the claim.

(4) $A/I$ is complete.
Consider a Cauchy sequence $(a_n + I)$ in $A/I$, so that $|a_{n+1} - a_n + I|_{A/I} \to 0$. Use the definition of $|\cdot|_{A/I}$ to pick recursively a sequence $(b_n)$ in $I$ with $b_0 = 0$ such that $|(a_{n+1} + b_{n+1}) - (a_n - b_n)|_A \to 0$. Then $(a_n + b_n)$ is a Cauchy sequence in $A$; since $A$ is complete, this sequence has a limit $a$. But then $|a_n - a + I|_{A/I} \le |(a_n + b_n) - a|_A \to 0$, so $(a_n + I)$ converges to $a + I$ in $A/I$.

(5) $A \to A/I$ is continuous.
This follows from $|(a + I) - (b + I)|_{A/I} \le |a - b|_A$. ❑

6.8. **Definition.** A $K$-Banach algebra $A$ is a *$K$-affinoid algebra* if there is a generalized Tate algebra $T$ and a surjective $K$-algebra homomorphism $\phi\colon T \to A$ such that the norm of $A$ is (equivalent to) the induced norm on $T/\ker(\phi) \cong A$. If $T$ can be taken to be a Tate algebra, then $A$ is a *strict $K$-affinoid algebra*.

**DEF**
affinoid algebra and domain

The space $\operatorname{Berk} A$ associated to a (strict) affinoid algebra is a *(strict) affinoid domain*. ♢

Simple examples of affinoid domains are the closed Berkovich disks $\mathcal{D}(0, r)$, where $A$ is itself a (generalized) Tate algebra.

6.9. **Example.** Let $0 < r \le s$. Then we can consider the algebra

**EXAMPLE**
annuli

$$A = K\langle s^{-1}X, rX^{-1}\rangle := \frac{K\langle s^{-1}X, rY\rangle}{\langle XY - 1\rangle}$$

with its induced norm. Using the relation $XY = 1$, its elements can be written uniquely as the image of a series of the form

$$\sum_{m=1}^{\infty} a_{-m}Y^m + a_0 + \sum_{n=1}^{\infty} a_n X^n$$

with $s^n|a_n| \to 0$ as $n \to \infty$ and $r^m|a_m| \to 0$ as $m \to -\infty$. We can interpret this (via $Y = X^{-1}$ in $A$) as the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n X^n\,;$$

the conditions on the coefficients then are equivalent to saying that this series converges whenever we plug in some element $\alpha$ such that $r \le |\alpha| \le s$. For $K = \mathbb{C}_p$, we see that via the map $\mathbb{C}_p[x] \to A$ sending $x$ to $X$, $\operatorname{Berk} A$ gets identified with $\mathcal{A}(r, s) := \mathcal{D}(0, s) \setminus \mathcal{D}(0, r)^-$, which is a closed *Berkovich annulus*. ♣

Most of the structure of affinoid algebras is intrinsic. We state the following without proof.

6.10. **Theorem.** *Let $A$ be a $K$-affinoid algebra.*

**THM** Properties of affinoid algebras

(1) *Any two $K$-Banach algebra norms on $A$ are equivalent. In particular, the topology on $A$ induced by the norm is intrinsic, and the same is true for notions of boundedness.*
(2) *Any $K$-algebra morphism $A' \to A$ from another $K$-affinoid algebra is continuous (and hence a morphism of $K$-affinoid algebras).*

There are more properties similar to those of algebras of finite type in algebraic geometry. For example, if $A$ is $K$-affinoid and $A'$ is a $K$-algebra that is a finitely generated $A$-module, then $A'$ is also $K$-affinoid. Any $K$-affinoid algebra is a finitely generated module over some Tate algebra (this corresponds to the Noether normalization theorem).

One can then construct more general Berkovich $K$-analytic spaces by gluing affinoid domains. There are some technical issues here, because the gluing is along compact 'sub-affinoid domains' and not along open sets as one does in algebraic or differential geometry.

For example, for fixed $s > 0$ one can glue the closed disks of radius $r < s$ via the natural inclusions $\mathcal{D}(0,r) \subset \mathcal{D}(0,r')$ whenever $r < r' < s$ to obtain the open disk $\mathcal{D}(0,s)^-$. In a similar way, one gets open annuli $\mathcal{A}(r,s)^-$ (for $r < s$). Considering the affine line as 'the open disk of radius $\infty$', we obtain $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$. We get the projective line $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ by gluing two closed disks $\mathcal{D}(0,1)$ along the annulus $\mathcal{A}(1,1)$ as discussed in the last section.

We also have the following (see the exercises). As usual, $R$ denotes the valuation ring of $K$ and $k$ the residue field.

6.11. **Theorem.** *Let $A$ be a $K$-Banach algebra. Then*

**THM** Reduction map

$$A^\circ = \bigl\{a \in A : \{|a^n|_A : n \ge 0\} \text{ is bounded}\bigr\}$$

*is a closed subalgebra of $A$ that contains (the image of) $R$ and depends on the norm of $A$ only up to equivalence (and so is completely intrinsic when $A$ is affinoid). The set $A^{\circ\circ} = \{a \in A : a^n \to 0\}$ is an ideal of $A^\circ$, and the quotient $\tilde{A} = A^\circ/A^{\circ\circ}$ is a $k$-algebra functorially associated to $A$. If $A$ is strictly $K$-affinoid, then $\tilde{A}$ is a finitely generated $k$-algebra. If $\|\cdot\| \in \operatorname{Berk} A$, then the set $\{a \in A^\circ : \|a\| < 1\}$ is a prime ideal of $A^\circ$ containing $A^{\circ\circ}$; in this way we obtain a canonical reduction map $\rho_A \colon \operatorname{Berk} A \to \operatorname{Spec} \tilde{A}$. If $A$ is strictly $K$-affinoid, then $\rho_A$ is anti-continuous: preimages of closed sets are open and vice versa.*

For example, taking $A = \mathbb{C}_p\langle x\rangle$, we find $\tilde{A} = \bar{\mathbb{F}}_p[x]$, and the reduction map sends $\zeta_{0,1}$ to the generic point of $\mathbb{A}^1_{\bar{\mathbb{F}}_p} = \operatorname{Spec} \tilde{A}$ and the points in the open Berkovich disk $\mathcal{D}(\alpha,1)^-$ (for $\alpha \in D(0,1)$) are mapped to the closed point $\bar{\alpha} \in \mathbb{A}^1(\bar{\mathbb{F}}_p)$. In particular, we see that the preimage of a closed point is an open Berkovich disk, and the preimage of the open set $\mathbb{A}^1_{\bar{\mathbb{F}}_p} \setminus S$, where $S$ is a finite set of closed points, is the closed set $\mathcal{D}(0,1) \setminus \bigcup_{s\in S} \mathcal{D}(\alpha_s,1)^-$, where $\alpha_s \in D(0,1)$ has image $s$.

6.12. **Example.** Let $0 < r < s$ be in $|K^\times|$ and consider as before the algebra $A = K\langle s^{-1}X, rX^{-1}\rangle$. The induced norm on $A$ is given by

**EXAMPLE** annuli

$$\Bigl|\sum_{n\in\mathbb{Z}} a_n X^n\Bigr|_A = \max\bigl(\{r^n|a_n| : n \le 0\} \cup \{s^n|a_n| : n \ge 0\}\bigr)$$

(this is because $|XY| = s/r > 1$, so the 'smallest' representative of $1 \in A = K\langle X,Y\rangle/\langle XY-1\rangle$ is 1). By choosing $c, C \in K$ such that $|c| = r$ and $|C| = s$ and scaling $X$ and $Y$, we can write $A$ as the quotient

$$A \cong K\langle X,Y\rangle/\langle XY - cC^{-1}\rangle$$

of the standard two-variable Tate algebra $T$. From this it is fairly easy to see that

$$\tilde{A} = \tilde{T}/\langle XY\rangle = k[X,Y]/\langle XY\rangle\,,$$

where $k$ is the residue field. So Spec $\tilde{A}$ is a union of two affine lines (corresponding to $X = 0$ and $Y = 0$) meeting transversally in one point ($X = Y = 0$). Let $\Gamma = [\zeta_{0,r}, \zeta_{0,s}] \subset \mathcal{A}(r,s)$. One can check that $\rho_A$ maps $\zeta_{0,r} \in \mathcal{A}(r,s)$ to the generic point of the line $X = 0$, $\zeta_{0,s}$ to the generic point of the line $Y = 0$, points retracting under $\tau_\Gamma$ to $\zeta_{0,r}$ to the point on $X = 0$ corresponding to their tangent direction and similarly for points retracting to $\zeta_{0,s}$, and all points retracting to any other point of $\Gamma$ (in the 'open interval' between the endpoints) to the point of intersection of the two lines. ♣

The anti-continuity of the reduction map $\rho_A$ implies that the gluing together of strict affinoids along closed sub-affinoids induces a corresponding gluing of the Spec $\tilde{A}$'s along open sets. For example, gluing two copies of $\mathcal{D}(0,1)$ along $\mathcal{A}(1,1)$ to obtain $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ corresponds to gluing to copies of $\mathbb{A}^1_{\bar{\mathbb{F}}_p}$ along $\mathbb{A}^1 \setminus \{0\}$ (which is Spec $\tilde{A}$ for the annulus $\mathcal{A}(1,1)$), resulting in $\mathbb{P}^1_{\bar{\mathbb{F}}_p}$ in the usual way. The reduction maps also glue, which in the example gives the map identifying the tangent directions at $\zeta_{0,1} \in \mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ with $\mathbb{P}^1(\bar{\mathbb{F}}_p)$ and sending $\zeta_{0,1}$ to the generic point of $\mathbb{P}^1_{\bar{\mathbb{F}}_p}$.

A given space can usually be obtained in many different ways by gluing. This can result in different 'special fibers' (the glued spectra of the algebras $\tilde{A}$) and therefore different reduction maps. For example, we can obtain $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ also by gluing $\mathcal{D}(0,1)$ (via the inversion map) and $\mathcal{D}(0,1/p)$ to $\mathcal{A}(1/p,1)$ along the sub-annuli $\mathcal{A}(1,1)$ and $\mathcal{A}(1/p,1/p)$. The special fiber of this object consists of two copies of $\mathbb{P}^1_{\bar{\mathbb{F}}_p}$ meeting transversally in one point. It is obtained from the special fiber of the annulus — two $\mathbb{A}^1$'s meeting in one point — by gluing an $\mathbb{A}^1$ to each of the lines (along the complement of the intersection point). In a similar way, we can glue together also more general affinoids (the most general of which is a closed disk minus a finite union of open ones) to obtain a 'model' of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$. The skeleton associated to such a model is the convex hull of the points mapping to generic points of the special fiber; in the preceding example, this would be the interval $[\zeta_{0,1/p}, \zeta_{0,1}]$. In general, it will be a finite sub-tree of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ with endpoints of type 2.

There is a converse to this, valid for general smooth projective irreducible curves $C$ over $\mathbb{C}_p$: There is a bijective correspondence between finite subgraphs of Berk $C$ with vertices of type 2 and semistable models of $C$ over the valuation ring $R$ of $\mathbb{C}_p$ (up to isomorphism), such that the vertices correspond to the generic points of the special fiber of the semistable model under the reduction map.[2]

[2] M. Baker, S. Payne, J. Rabinoff, *On the structure of non-Archimedean analytic curves.* Tropical and non-Archimedean geometry, 93–121, Contemp. Math., **605**, Amer. Math. Soc., Providence, RI, 2013; arXiv: 1404.0279.

## 7. Berkovich spaces of curves

In this section we will take a closer look at Berkovich spaces of (smooth projective irreducible) curves over $\mathbb{C}_p$. We begin with some lemmas. Let $T_r = \mathbb{C}_p\langle r^{-1}x\rangle$ be the univariate (generalized) Tate algebra with parameter $r > 0$.

7.1. **Lemma.** *Let*

**LEMMA** roots of polynomials over Tate algebras

$$F(x,y) = F_n(x)y^n + F_{n-1}(x)y^{n-1} + \ldots + F_0(x) \in T_r[y]$$

*be a polynomial such that there is $\delta < 1$ with $|F_0|_r \le \delta$, $|1 + F_1|_r \le \delta$ and $|F_j|_r \le \delta^{2-j}$ for all $j \ge 2$. Then there is $h(x) \in T_r$ such that $|h(0)|_p < 1$ and $F(x, h(x)) = 0$.*

*Proof.* The equation $F(x,y) = 0$ is equivalent to the fixed point equation

$$y = \Psi(y) := F_0(x) + (1 + F_1(x))y + F_2(x)y^2 + \ldots + F_n(x)y^n .$$

For $|y_1|, |y_2| \le \delta$ we have

$$\begin{aligned}|\Psi(y_2) - \Psi(y_1)|_r &= |y_2 - y_1|_r \\ &\quad \cdot |(1 + F_1(x)) + F_2(x)(y_2 + y_1) + \ldots + F_n(x)(y_2^{n-1} + y_2^{n-2}y_1 + \ldots + y_1^{n-1})|_r \\ &\le \max\{|1 + F_1|_r, |F_2|_r\delta, \ldots, |F_n|_r\delta^{n-1}\} \cdot |y_2 - y_1|_r \\ &\le \delta \cdot |y_2 - y_1|_r .\end{aligned}$$

Also,

$$|\Psi(y)|_r \le \max\{|F_0|_r, |1 + F_1|_r\delta, |F_2|_r\delta^2, \ldots\} \le \delta$$

when $|y|_r \le \delta$. So $\Psi$ defines a contracting map on $\{y \in T_r : |y|_r \le \delta\}$. The Banach Fixed Point Theorem then gives us a unique solution. ❑

7.2. **Lemma.** *Let $F(x,y) \in T_r[y]$ be monic of degree $n$ such that $F(0,y) \in \mathbb{C}_p[y]$ has no multiple roots. Then there is $0 < r' \le r$ and there are $h_1, \ldots, h_n \in T_{r'}$ such that $F(x,y) = (y - h_1(x)) \cdots (y - h_n(x))$.*

**LEMMA** étale covering of disk

*Proof.* Let $\eta_1, \ldots, \eta_n \in \mathbb{C}_p$ be the $n$ distinct roots of $F(0,y)$. By assumption, $\beta_j := F'(0,\eta_j) \neq 0$ (where $F'$ denotes the derivative of $F$ as a polynomial in $y$). Let $\tilde{F}_j(x,y) := -\beta_j^{-1}F(x, y + \eta_j)$ Note that $\tilde{F}_j$ has no constant term and that the coefficient of $y$ is $-1$. So, writing $\tilde{F}_j(x,y) = f_0(x) + f_1(x)y + \ldots$ and setting $\gamma = \max\{|f_0|_r, |1 + f_1|_r, |f_2|_r, \ldots\}$, we have for every $0 < r' \le r$ that

$$|f_0|_{r'} \le \gamma r'/r\,, \quad |1 + f_1(x)|_{r'} \le \gamma r'/r \quad \text{and} \quad |f_m|_{r'} \le \gamma \quad \text{for } m \ge 2.$$

Now pick $\lambda \in \mathbb{C}_p^\times$ with $|\lambda|_p \le \gamma^{-1}$ and set

$$\tilde{\tilde{F}}_j(x,y) = \lambda^{-1}\tilde{F}_j(x, \lambda y) = \lambda^{-1}f_0(x) + f_1(x)y + \lambda f_2(x)y^2 + \ldots ;$$

we have $|\lambda^{-1}f_0|_{r'} \le |\lambda|_p^{-1}\gamma r'/r$, $|1+f_1|_{r'} \le \gamma r'/r$ and $|\lambda^{m-1}f_m|_{r'} \le |\lambda|_p^{m-1}\gamma \le \gamma^{2-m}$ for $m \ge 2$. Fix $0 < \delta < 1$ such that $\delta \le \gamma^{-1}$. With $r' = r\min\{1, |\lambda|_p\gamma^{-1}\delta, \gamma^{-1}\delta\}$ the assumptions of Lemma 7.1 are then satisfied for $\tilde{\tilde{F}}_j$, so (translating back to the original coordinates) there is $h_j \in T_{r'}$ with $h_j(0) = \eta_j$ and $F(x, h_j(x)) = 0$. We can take $r'$ so that it works for all $j$; we then obtain $n$ distinct roots of $F$ in $T_{r'}$. ❑

I think that one can take any $r' \le r$ here such that $F(\xi, y)$ has no multiple roots for all $\xi \in \mathbb{C}_p$ with $|\xi|_p \le r'$, at least when $p > n$, but so far I have found no proof for this more precise statement. We will see below that it is true when $n = 2$ and $p > 2$.

7.3. **Example.** The example $F(x, y) = y^2 - y + x$ over $\mathbb{C}_2$ shows that the condition $p > n$ is necessary in general. The two roots of $F$ are

**EXAMPLE** necessity of '$p > n$'

$$h_1 = \frac{1 - \sqrt{1 - 4x}}{2} = x + x^2 + 2x^3 + 5x^4 + 14x^5 + \ldots = \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n$$

and $h_2 = 1 - h_1$. The coefficients are integers, but the coefficient of $x^{2^n}$ is odd for all $n$, so the coefficients of the series are bounded but do not tend to zero 2-adically. This means that $h_1, h_2 \in T_r$ for all $r < 1$, but not for $r = 1$. On the other hand, the discriminant of $F$ is $1 - 4x$, which does not vanish even for $|x|_2 < 4$. This seems to be some issue related to 'wild ramification'. ♣

7.4. **Theorem.** *Let $C$ be a smooth projective irreducible curve over $\mathbb{C}_p$ and let $\phi\colon C \to \mathbb{P}^1$ be a morphism of degree $n$. Then the induced map $\phi_*\colon C^{\mathrm{an}} \to \mathbb{P}^{1,\mathrm{an}}$ has fibers of size at most $n$.*

**THM** coverings of $\mathbb{P}^1$

*Proof.* Let $\zeta \in \mathbb{P}^{1,\mathrm{an}}$ be some point; we can assume without loss of generality that $\zeta \neq \infty$. The statement is clear when $\zeta$ is a type 1 point, so we consider $\zeta$ not of type 1; in particular, the seminorm on $\mathbb{C}_p[x]$ associated to $\zeta$ is a norm. The point $\zeta$ also corresponds to a homomorphism $\mathbb{C}_p[x] \to \mathcal{H}$ into a complete field $\mathcal{H}$ with absolute value such that the absolute value pulls back to the seminorm given by $\zeta$ and the image of $\mathbb{C}_p[x]$ in $\mathcal{H}$ generates a dense subfield. Let $K$ be the function field of $C$; we have an inclusion $\mathbb{C}_p(x) \subset K$, which is a field extension of degree $n$. Then $\mathcal{H} \otimes_{\mathbb{C}_p(x)} K$ is an algebra over $\mathcal{H}$ of degree $n$, which splits as a direct product of finite field extensions of $\mathcal{H}$ (whose degrees sum to $n$). There is a unique extension of the absolute value of $\mathcal{H}$ to each of these fields, which by pulling back to $K$ (and then restricting to the affine coordinate ring $A$ of $\phi^{-1}(\mathbb{A}^1)$) give rise to the various multiplicative (semi)norms on $A$ extending $\|\cdot\|_\zeta$. So the fiber $\phi_*^{-1}(\zeta)$ has as many points as there are fields in the splitting of $\mathcal{H} \otimes_{\mathbb{C}_p(x)} K$; this number is at most $n$. ❑

This basically shows that we can glue together $C^{\mathrm{an}}$ from $n$ copies of $\mathbb{P}^{1,\mathrm{an}}$. If the statement above on the choice of $r'$ is true and $p > n$, then it follows that $\phi_*$ can be branched (i.e., have fibers of size $< n$) only above points in the convex hull of the type 1 points corresponding to branch points of $\phi$. This might be true even when this statement is wrong; in any case it would mean that $C^{\mathrm{an}}$ deformation retracts to a finite graph that is glued from $n$ copies of a tree (= the convex hull of the branch points).

We now consider hyperelliptic curves in more detail. We assume that $p > 2$. First we need two elementary lemmas.

7.5. **Lemma.** *Let $f(x) \in T_r$ with $|f|_r < 1$. Then there is $h(x) \in T_r$ with $|h|_r < 1$ such that $1 + f(x) = (1 + h(x))^2$.*

**LEMMA** square roots

*More generally, the same result applies to any complete ring $R$ in place of $T_r$ as long as $|2|_R = 1$ and $2 \in R^\times$.*

*Proof.* We want a root $h$ of $F(x, y) = -\frac{1}{2}y^2 - y + \frac{1}{2}f(x)$ with $|h - 1|_r < 1$. Since $|2|_p = 1$ and $|f|_r < 1$, we can directly apply Lemma 7.1. The more general statement follows in the same way. ❑

7.6. **Lemma.** *Let $\mathcal{H}$ be the completion of the field of fractions of $T_r$. Then $x$ is not a square in $\mathcal{H}$.* **LEMMA** $x$ not a square

*Proof.* First consider the case that $r \in |\mathbb{C}_p^\times|$. Then we can assume $r = 1$, since $T_r$ is isomorphic to $T_1$ and the isomorphism only scales $x$. Assume that $x = z^2$ is a square in $\mathcal{H}$. Since the field of fractions of $T_1$ is dense in $\mathcal{H}$, there must be $a, b \in T_1$ such that $|z - a/b|$ is small, which implies that $|a(x)^2 - xb(x)^2| \ll |b(x)|^2$. We can scale $a$ and $b$ so that $|b| = 1$; then also $|a| \le 1$. Looking at the reductions, we must have that $a(x)^2 - xb(x)^2$ reduces to zero in $\bar{\mathbb{F}}_p[x]$, but this is impossible, since the first term has even degree and the second term is nonzero and has odd degree.

To deal with arbitrary $r$, we can enlarge $\mathbb{C}_p$ to obtain a field $K$ (that is again complete and algebraically closed) such that $r \in |K^\times|$. Then the argument above shows that $x$ is not even a square in a larger field. ❑

The preceding lemma can be generalized: if $f \in T_1^\circ$ has reduction in $\bar{\mathbb{F}}_p[x]$ that is not a square, then $f$ is not a square in $\mathcal{H}$. Conversely (assuming $p$ is odd as we do here), one can show using Lemma 7.5 that when the reduction of a polynomial $f \in \mathbb{C}_p[x] \cap T_1^\circ$ is a nonzero square, then $f$ is a square in $\mathcal{H}$. See the exercises.

A hyperelliptic curve $C$ of genus $g$ over $\mathbb{C}_p$ can be glued together from two affine pieces, the plane curves

$$y^2 = f_{2g+2}x^{2g+2} + f_{2g+1}x^{2g+1} + \ldots + f_1 x + f_0$$

and

$$Y^2 = f_0 X^{2g+2} + f_1 X^{2g+1} + \ldots + f_{2g+1}X + f_{2g+2}$$

with the identifications $xX = 1$ and $Y = X^{g+1}y$; we assume that the polynomial $f(x) = \sum_j f_j x^j$ has no multiple roots and degree at least $2g + 1$. If we assume that $f_{2g+2} \neq 0$, then $C \to \mathbb{P}^1_x$ is unbranched above infinity, so we can restrict our attention to the first affine patch. We denote the double cover $C \to \mathbb{P}^1$ given by the $x$-coordinate by $\pi$, and we denote by $\Theta$ the set of branch points in $\mathbb{P}^1(\mathbb{C}_p)$ (i.e., the roots of $f$), which we also consider as type 1 points of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$.

7.7. **Lemma.** *Let $\zeta = \zeta_{0,r} \in \mathbb{A}^{1,\mathrm{an}}$ be of type 2 or 3. We have a natural partition of $\Theta$ into finitely many nonempty subsets $\Theta_v$ (indexed by the tangent direction $v$ in which the points in $\Theta_v$ are visible). Then $\pi_*^{-1}(\zeta_{0,r})$ consists of two points if all subsets $\Theta_v$ have even cardinality, and of one point otherwise.* **LEMMA** branching of $\pi$

*Proof.* If $r \notin |\mathbb{C}_p^\times|$ (then $\zeta$ is of type 3), then there are just two tangent directions, which we denote 0 and $\infty$. Otherwise we select one point $a \in \mathbb{P}^1(\mathbb{C}_p)$ representing each occurring tangent direction different from 'up' and write $\Theta_a$ for the corresponding subset (and $\Theta_\infty$ for the subset of roots $\xi$ such that $|\xi|_p > r$). Then we can write

$$f(x) = c \prod_{\xi \in \Theta_\infty} \Bigl(1 - \frac{x}{\xi}\Bigr) \cdot \prod_a \Bigl((x-a)^{\#\Theta_a} \prod_{\xi \in \Theta_a} \Bigl(1 - \frac{\xi - a}{x - a}\Bigr)\Bigr)$$

with some $c \in \mathbb{C}_p^\times$. By Lemma 7.5 (note that $|x/\xi|_r < 1$ when $|\xi|_p > r$) each factor $1 - x/\xi$ in the first product is a square in $T_r$. Now let $\mathcal{H}$ be the completion of the field of fractions of $T_r$. Then for $\xi \in \Theta_a$, we have $|\xi - a|_p < r$ and $|x - a|_r = r$, so $(\xi - a)/(x - a) \in \mathcal{H}$ has absolute value $< 1$, so $1 - (\xi - a)/(x - a)$ is also a square in $\mathcal{H}$ using Lemma 7.5 again. Since $\mathbb{C}_p$ is algebraically closed, $c$ is of course also a square. So $f(x)$ is a square in $\mathcal{H}$ times $\prod_a (x - a)^{\#\Theta_a}$. By Lemma 7.6 and the remark following it, it follows that the latter and hence $f(x)$ is a square in $\mathcal{H}$ if and only if all exponents $\#\Theta_a$ are even (if $\zeta$ is of type 3, then we only need that $x$ is not a square; if $\zeta$ is of type 2, we can assume that $r = 1$). On the other hand, the degree 2 algebra over $\mathcal{H}$ induced by $\pi$ is $\mathcal{H}[y]/\langle y^2 - f(x)\rangle$. This is a field if and only if $f(x)$ is not a square in $\mathcal{H}$; otherwise it splits as a product of two copies of $\mathcal{H}$ (by the Chinese Remainder Theorem). Since the number of fields in the product decomposition of this algebra is the number of points in the fiber of $\pi_*$ above $\zeta$, this completes the proof. ❑

7.8. **Corollary.** *Let $p > 2$ and let $C\colon y^2 = f(x)$ be a hyperelliptic curve over $\mathbb{C}_p$ with hyperelliptic double cover $\pi\colon C \to \mathbb{P}^1$. Then the induced double cover $\pi_*\colon C^{\mathrm{an}} \to \mathbb{P}^{1,\mathrm{an}}$ is branched exactly above points $\zeta$ such that there is at least one tangent direction at $\zeta$ that points to an odd number of roots of $f$. In particular, branching only occurs along the convex hull $\mathcal{T}$ of the branch points of $\pi$ (i.e., the roots of $f$) considered as points of type 1, and $C^{\mathrm{an}}$ can be obtained by gluing two copies of $\mathbb{P}^{1,\mathrm{an}}$ along parts of $\mathcal{T}$.*

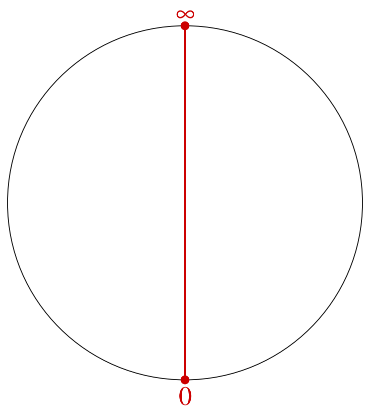
**COR** hyperelliptic Berkovich curve

*Proof.* Let $\zeta \in \mathbb{P}^{1,\mathrm{an}}$. If $\zeta \notin \mathcal{T}$, then there is some Berkovich disk $\mathcal{D}(a, r)$ containing $\zeta$ such that $D(a, r)$ does not contain branch points of $\pi$. The proof of Lemma 7.7 in the case that $\Theta = \Theta_\infty$ (or also Lemma 7.5 directly) then shows that $f(x+a)$ is a square in $\mathbb{C}_p\langle r^{-1}x\rangle$, which implies that $\pi_*$ is unramified above $\mathcal{D}(a, r)$ and in particular above $\zeta$. So branching can only occur along $\mathcal{T}$; in particular, no branching occurs at points of type 4 (which is consistent with the statement, since there is only one tangent directions in which all $2g+2$ branch points can be seen). For points $\zeta_{a,r} \in \mathcal{T}$ not of type 1, the statement on branching is Lemma 7.7 (after shifting $x$ by $a$). For the points $\zeta$ of type 1, the statement is clear — note that when $\zeta$ is a branch point of $\pi$, then one can see the *other* $2g + 1$ branch points in the unique tangent direction; otherwise one sees all $2g + 2$ branch points in this direction. ❑

Since $\mathbb{P}^{1,\mathrm{an}}$ deformation retracts to $\mathcal{T}$, we see that $C^{\mathrm{an}}$ will deformation retract to what is obtained by gluing two copies of the tree $\mathcal{T}$ along the set of points that partition the set of branch points (which are the leaves of $\mathcal{T}$) into subsets at least one of which has an odd number of elements. For simplicity, let us say that a vertex or edge of $\mathcal{T}$ is *even*, if the partition of the leaves it induces results in sets with even cardinality, and *odd* otherwise. We will consider leaves as odd vertices. Then the edge connecting a leaf to its adjacent vertex is always odd, so (as long as there is at least one non-leaf vertex of $\mathcal{T}$) at the cost of another deformation retraction, we can remove these edges from the graph obtained by gluing the trees. We can continue this process as long as there are terminal odd edges (and at least one vertex remains). In particular, we see that $C^{\mathrm{an}}$ can be contracted to a point (and so is simply connected) if and only if *all* edges are odd (equivalently, the branch locus is all of $\mathcal{T}$). Otherwise we can find two odd vertices connected by a chain of even edges and vertices, and the gluing produces a circle, so the resulting space is not simply connected.

**EXAMPLES** double covers of $\mathbb{P}^{1,\mathrm{an}}$

7.9. **Examples.** The simplest case is when $g = 0$ (strictly speaking, a double cover of $\mathbb{P}^1$ is a hyperelliptic curve only when $g \geq 2$, but we can consider them also for $g = 0$ or $g = 1$). Up to a change of coordinates, we can take $f(x) = x$; then the branch locus in $\mathbb{P}^1(\mathbb{C}_p)$ consists of the two points 0 and $\infty$. Their convex hull $\mathcal{T}$ is simply the arc connecting the two (it consists of all points $\zeta_{0,r}$ for $r \geq 0$ together with $\infty$). So as a tree, $\mathcal{T}$ has two leaves connected by an odd edge. By the remarks above, $C^{\mathrm{an}}$ is simply connected, which is no surprise, since $C \cong \mathbb{P}^1$. In the sketches below, $\mathbb{P}^1(\mathbb{C}_p)$ is symbolized by the circle and $\mathbb{P}^{1,\mathrm{an}}$ by the disk it encloses; odd vertices and edges of $\mathcal{T}$ are red and even ones (which do not exist here) are green.




The next case is when $g = 1$; it is a bit more interesting. We have four branch points; by a change of coordinates, we can move three of them to 0, 1, $\infty$. Then $f(x) = x(x-1)(x-\lambda)$ for some $\lambda \in \mathbb{C}_p \setminus \{0, 1\}$. There are four automorphisms of $\mathbb{P}^1$ that fix a set of four points, but there are 24 ways of selecting the three points that are mapped to 0, 1, $\infty$, so each curve $C$ arises from (in general) six different choices of $\lambda$. If $\lambda$ is one of them, then the others are

$$1 - \lambda\,, \quad \frac{1}{\lambda}\,, \quad 1 - \frac{1}{\lambda}\,, \quad \frac{1}{1-\lambda} \quad \text{and} \quad \frac{\lambda}{\lambda - 1}\,.$$

In particular, we can assume that $|\lambda|_p \leq 1$. There are now two cases.

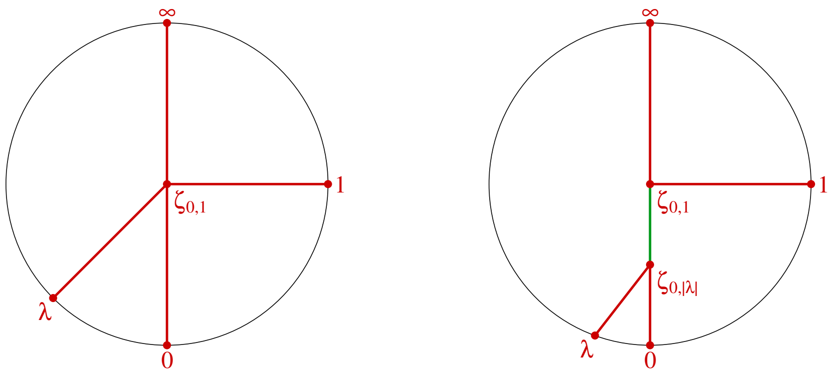
If $|\lambda|_p = |1-\lambda|_p = 1$, then the four branch points lie in four different tangent directions at $\zeta_{0,1}$, since the reduction $\bar{\lambda} \in \mathbb{P}^1(\bar{\mathbb{F}}_p)$ is distinct from (the reductions of) 0, 1, $\infty$. So the tree $\mathcal{T}$ has one inner vertex of degree 4 and four edges connecting it to the four leaves. In particular, all vertices and edges are odd, and $C^{\mathrm{an}}$ is simply connected. In this case $C$ is an elliptic curve with good reduction.

In the second case, we can assume (perhaps after replacing $\lambda$ by $1 - \lambda$) that $|\lambda|_p < 1$. Then at $\zeta_{0,1}$, we see one branch point each when looking to $\infty$ or 1, but we see two (namely 0 and $\lambda$) when looking to 0. We get a similar partition at $\zeta_{0,|\lambda|_p}$, which is now $\{0\}$, $\{\lambda\}$, $\{1, \infty\}$. So $\mathcal{T}$ has the two (odd) inner vertices $\zeta_{0,1}$ and $\zeta_{0,|\lambda|_p}$, each of degree 3, there is one (even) edge connecting them, and each of the two vertices has two more (odd) edges connecting them to two of the branch points (1 and $\infty$, resp., 0 and $\lambda$). Gluing two copies of $\mathcal{T}$ along the odd edges and vertices then gives a graph that looks like $\mathcal{T}$ with the central edge doubled. This double edge forms a circle to which everything else can be retracted. Its length in the big metric is twice the length of the edge, so it is $2v_p(\lambda)$, which is the same as

$-v_p(j(C))$, where $j(C)$ is the $j$-invariant of the elliptic curve $C$, given by

$$j(C) = 2^8 \, \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1-\lambda)^2} \, .$$

In any case, $C^{\mathrm{an}}$ is not simply connected; its fundamental group is that of the circle, so $\cong \mathbb{Z}$. Taking the universal covering space 'unwraps' the circle and replaces it by a line; the universal covering $\tilde{C}^{\mathrm{an}}$ of $C^{\mathrm{an}}$ is $\mathbb{G}_{\mathrm{m}}^{\mathrm{an}} = \mathbb{P}^{1,\mathrm{an}} \setminus \{0, \infty\}$. ($\mathbb{G}_{\mathrm{m}}$ denotes the multiplicative group; its affine coordinate ring is $\mathbb{C}_p[x, x^{-1}]$ and its $\mathbb{C}_p$-points are $\mathbb{G}_{\mathrm{m}}(\mathbb{C}_p) = \mathbb{C}_p^\times$.) Tate has shown (using different language) that the covering map $\mathbb{G}_{\mathrm{m}}^{\mathrm{an}} \to C^{\mathrm{an}}$ is actually a group homomorphism and that the group of deck transformations is generated by 'multiplication by $q$', where $q \in \mathbb{C}_p^\times$ satisfies $|q|_p < 1$. Pulling back the coordinate functions to $\mathbb{G}_{\mathrm{m}}$ gives an explicit analytic uniformization $C(\mathbb{C}_p) \cong \mathbb{C}_p^\times / q^{\mathbb{Z}}$ that is similar in spirit to the uniformization over the complex numbers: $C(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \cong \mathbb{C}^\times / q^{\mathbb{Z}}$, where the second isomorphism is induced by $z \mapsto e^{2\pi i z}$ and $q = e^{2\pi i \tau}$, with $\tau \in \mathbb{C}$ in the upper half-plane (so $|q| < 1$).




Left: $|\lambda|_p = |1-\lambda|_p = 1$, right: $|\lambda|_p < 1$. ♣

## 8. Integration

In this final section we want to consider integration on Berkovich spaces. The goal is to have a 'reasonable' way of associating to a closed 1-form $\omega$ (meaning that $d\omega = 0$) on an analytic space $X$ and a path $\gamma\colon [0,1] \to X$ whose endpoints are 'type 1 points' (i.e., they correspond to seminorms whose kernel is a maximal ideal, so functions can be evaluated at such points to give a value in $\mathbb{C}_p$) an integral

$$\int_\gamma \omega \in \mathbb{C}_p \,.$$

By 'reasonable', we mean that this integral has the properties we expect from such a notion, for example:

(1) **Linearity in the 1-form.** $\displaystyle\int_\gamma (\alpha_1\omega_1 + \alpha_2\omega_2) = \alpha_1 \int_\gamma \omega_1 + \alpha_2 \int_\gamma \omega_2$

for $\alpha_1, \alpha_2 \in \mathbb{C}_p$ and $\omega_1, \omega_2$ closed 1-forms on $X$.

(2) **Additivity on paths.** $\displaystyle\int_{\gamma_1+\gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$

when $\gamma_1(1) = \gamma_2(0)$ and $\gamma_1 + \gamma_2$ is the path obtained by traversing first $\gamma_1$ and then $\gamma_2$.

(3) **Fundamental theorem of calculus.** $\displaystyle\int_\gamma df = f(\gamma(1)) - f(\gamma(0))$

when $f$ is a function on $X$.

(4) **Change of variables.** $\displaystyle\int_\gamma \phi^*\omega = \int_{\phi\circ\gamma} \omega$

when $\phi\colon Y \to X$ is a morphism, $\gamma$ is a path in $Y$ and $\omega$ is a closed 1-form on $X$.

(5) **Homotopy invariance.** $\displaystyle\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$

when $\gamma_1$ and $\gamma_2$ are homotopic with fixed endpoints. (This includes invariance under orientation-preserving re-parameterization of the path.)

We note that Property (3) uniquely determines the integral when $X = \mathcal{D}(0,1)^-$ is an open disk, since then any 1-form on $X$ has the form $\omega = f(t)\,dt$ where $f(t) = \sum_{n=0}^\infty a_n t^n$ converges on $D(0,1)^-$, and

$$f(t)\,dt = d\sum_{n=1}^\infty \frac{a_{n-1}}{n} t^n \,,$$

where the series on the right converges, too (since for $0 \le r < 1$ we have that $r^n/|n|_p \le nr^n \to 0$). So every 1-form is exact and Property (3) applies. The same is true for open Berkovich 'poly-disks' (which are the analytic spaces associated to a product of open disks in $\mathbb{C}_p$), when we deal with higher-dimensional spaces.

As a more interesting example, let us consider the logarithm, which should be defined on $X = \mathbb{G}_{\mathrm{m}}^{\mathrm{an}} = \mathbb{P}^{1,\mathrm{an}} \setminus \{0, \infty\}$ by the integral

$$\log x = \int_1^x \frac{dt}{t} \,.$$

Recall that $\mathbb{G}_m^{an}$ is simply connected, so by Property (5) the integral depends only on the endpoints and not on the path (which is basically unique here anyway). If $|x-1|_p < 1$, then we can apply Property (3) using Property (4) for the inclusion $\mathcal{D}(1,1)^- \subset \mathbb{G}_m^{an}$; this leads to the usual expression

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \mp \ldots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n} \qquad \text{for } |z|_p < 1.$$

We can do the same within any open disk not containing zero; using Property (2) this gives us

$$\log\bigl(a(1+z)\bigr) = \log a + \log(1+z) \qquad \text{for } a \in \mathbb{C}_p^\times,\ |z|_p < 1.$$

The question then is, how to fix the values of $\log a$ when $|a-1|_p \ge 1$?

If we use another important property of the logarithm, namely its functional equation

$$\log(xy) = \log x + \log y\,,$$

then this determines log on the subgroup $R^\times = \{a \in \mathbb{C}_p^\times : |a|_p = 1\}$: If $a \in R^\times$, then there is a root of unity $\zeta \in R^\times$ such that $|a-\zeta|_p < 1$. The functional equation forces us to set $\log\zeta = 0$, so

$$\log a = \log\zeta + \log(\zeta^{-1}a) = \log(1+z)$$

with $|z|_p = |\zeta^{-1}a - 1|_p < 1$. Note that the functional equation follows from the properties an integral should have:

$$\log(xy) = \int_1^{xy}\frac{dt}{t} \overset{(2)}{=} \int_1^{x}\frac{dt}{t} + \int_x^{xy}\frac{dt}{t} \overset{(4)}{=} \int_1^{x}\frac{dt}{t} + \int_1^{y}\frac{dt}{t} = \log x + \log y\,,$$

where we use that $dt/t$ is invariant under the scaling map $t \mapsto xt$.

This does not tell us, however, how to define $\log p$, say. In fact, it turns out that we can give $\log p$ an arbitrary value $\lambda$, but then log is indeed uniquely determined as

$$\log(up^v) = \log u + v\lambda \qquad \text{for } |u|_p = 1,\ v \in \mathbb{Q}.$$

We can take $\lambda = 0$, or (at the other extreme) treat $\lambda$ as an indeterminate and consider our integrals to have values in $\mathbb{C}_p[\lambda]$. We write $\log^\lambda$ for the 'branch of the logarithm' with $\log^\lambda p = \lambda$.

As it turns out, fixing $\lambda$ already fixes the integral. The following was proved by Berkovich [Ber, Thm. 9.1.1].

8.1. **Theorem.** *If we fix $\lambda$, then there is a unique integral satisfying properties (1) to (5) and* **THM** Existence and uniqueness of integral

(6) $$\int_1^p \frac{dt}{t} = \lambda.$$

(In fact, Berkovich does more: he defines sheaves of functions on $X$ that allow for iterated integration.)

8.2. **Example.** Consider the open Berkovich annulus $A = \mathcal{A}(r, R)^-$. In a similar way as for an open disk, the analytic functions on $A$ can be written as Laurent series

**EXAMPLE** Integral on an annulus

$$f(t) = \ldots + a_{-3}t^{-3} + a_{-2}t^{-2} + a_{-1}t^{-1} + a_0 + a_1 t + a_2 t^2 + \ldots = \sum_{n=-\infty}^{\infty} a_n t^n$$

that converge for all $\tau \in \mathbb{C}_p$ such that $r < |\tau|_p < R$. Then $f(t)\,dt = dg(t) + a_{-1}dt/t$ where

$$g(t) = \sum_{n \neq 0} \frac{a_{n-1}}{n} t^n$$

is again an analytic function on $A$. Recall that $A$ is simply connected, so integrals depend only on the endpoints. If $\tau_1, \tau_2 \in C_p$ satisfy $r < |\tau_j|_p < R$, then we obtain

$$\int_{\tau_1}^{\tau_2} f(t)\,dt = \int_{\tau_1}^{\tau_2} \Big(dg(t) + a_{-1}\frac{dt}{t}\Big) \overset{(1)}{=} \int_{\tau_1}^{\tau_2} dg(t) + a_{-1}\int_{\tau_1}^{\tau_2} \frac{dt}{t}$$
$$\overset{(3,4,6)}{=} g(\tau_2) - g(\tau_1) + a_{-1}\log^{\lambda}\frac{\tau_2}{\tau_1}\,.$$

Here we use the change of variables formula for the map $A \to \mathbb{G}_{\mathrm{m}}^{\mathrm{an}}$ that multiplies by $\tau_1^{-1}$ and (6). So if we want the integral to be well-defined (i.e., independent of the choice of $\lambda$), we need $a_{-1} = 0$, so that $f(t)\,dt = dg(t)$ is exact. ♣

8.3. **Example.** Consider an elliptic curve $E$ over $\mathbb{C}_p$ (say, $p > 2$, but this is not really necessary) such that $v_p(j(E)) < 0$. Then (as Tate has shown) there is $q \in \mathbb{C}_p^\times$ such that $|q|_p < 1$ and $E(\mathbb{C}_p) \simeq \mathbb{C}_p^\times/q^{\mathbb{Z}}$; the latter extends to a covering map $\pi\colon \mathbb{G}_{\mathrm{m}}^{\mathrm{an}} \to \mathbb{G}_{\mathrm{m}}^{\mathrm{an}}/q^{\mathbb{Z}} \simeq E^{\mathrm{an}}$ that exhibits $\mathbb{G}_{\mathrm{m}}^{\mathrm{an}}$ as the universal covering of $E^{\mathrm{an}}$ (recall that $\mathbb{G}_{\mathrm{m}}^{\mathrm{an}}$ is simply connected). Let $0 \neq \omega \in \Omega^1(E)$ be a regular (and therefore invariant) differential on $E$ and let $\gamma\colon [0,1] \to E^{\mathrm{an}}$ be the closed path that is the image under $\pi$ of the (unique) path from 1 to $q$ in $\mathbb{G}_{\mathrm{m}}^{\mathrm{an}}$. Then $\pi^*\omega$ is an invariant differential on $\mathbb{G}_{\mathrm{m}}$, so $\pi^*\omega = \alpha\,dt/t$ for some $\alpha \in \mathbb{C}_p^\times$. Then

**EXAMPLE** Integral on Tate ell. curve

$$\int_\gamma \omega \overset{(4)}{=} \int_1^q \alpha\frac{dt}{t} \overset{(1)}{=} \alpha\int_1^q \frac{dt}{t} \overset{(6)}{=} \alpha\log^{\lambda} q\,.$$

There is a unique $\lambda \in \mathbb{C}_p$ such that $\log^{\lambda} q = 0$; if we choose this value, then integrals over $\omega$ on $E$ will not depend on the path. But this will not work for *two* Tate curves simultaneously when their $q$ parameters are not multiplicatively dependent. So we cannot avoid the path-dependence of the integral. In particular, there will not be a function $f$ (in the sense of Berkovich's integration theory) on all of $E^{\mathrm{an}}$ such that $df = \omega$. This is somewhat analogous with the situation over $\mathbb{C}$, where there is no holomorphic function $f$ on $\mathbb{C}^\times$ such that $df = dz/z$. ♣

There is a different kind of integral that one can define on elliptic curves, and more generally, abelian varieties over $\mathbb{C}_p$. We will do this here for regular (= invariant) 1-forms, but the theory can be extended to arbitrary algebraic 1-forms on analytic spaces associated to algebraic varieties, see [Col].

This is similar to the standard logarithm, which is related to the group structure on the multiplicative group $\mathbb{G}_{\mathrm{m}}$. Let $\omega$ be an invariant 1-form on an abelian variety $A$ over $\mathbb{C}_p$ (for example, an elliptic curve). There is an open (and closed) neighborhood $U$ of the origin $0 \in A(\mathbb{C}_p)$ that is a subgroup and analytically isomorphic to an open polydisk $D$. Every closed 1-form is exact on an open polydisk, so we obtain a function $\log_\omega \colon U \to \mathbb{C}_p$ that satisfies $\log_\omega x = \int_0^x \omega$ for $x \in U$. By the same argument as before, $\log_\omega$ is a homomorphism on $U$. Contrary to $\mathbb{G}_{\mathrm{m}}$, $A$ is projective, which implies that $A(\mathbb{C}_p)/U$ is a torsion group (every element has finite order), so in this case there is a *unique* extension of $\log_\omega$ to all of $A(\mathbb{C}_p)$ as a group homomorphism $\log_\omega \colon A(\mathbb{C}_p) \to \mathbb{C}_p$: if $x \in A(\mathbb{C}_p)$ is arbitrary, then there is some $n \in \mathbb{Z}_{\ge 1}$ such that $nx \in U$; we must then have $\log_\omega x = (\log_\omega(nx))/n$ (and this does not depend on the choice of $n$ with $nx \in U$). We define, for $\omega$ as above and $x, y \in A(\mathbb{C}_p)$,

$$^{\mathrm{Ab}}\!\!\int_x^y \omega = \log_\omega y - \log_\omega x \,.$$

Then this integral satisfies Properties (1), (2) and (4) for abelian varieties and invariant differentials, where in (4) we only consider morphisms as varieties (which are morphisms as abelian varieties composed with a translation). This integral depends only on the endpoints and does not involve a choice of path, so Property (5) is trivially satisfied. We can extend it to (the Berkovich spaces associated to) arbitrary smooth projective varieties and closed regular 1-forms as follows. Let $V$ be such a variety and let $A$ be its Albanese variety; write $[y - x]$ for the image of $(y, x)$ under the Albanese map $V \times V \to A$. Fixing $x_0 \in V$, set $\alpha \colon V \to A$, $x \mapsto [x - x_0]$. Then if $\omega$ is a closed regular 1-form on $V$, there is an invariant 1-form $\omega_A$ on $A$ such that $\alpha^*\omega_A = \omega$ (this does not depend on the choice of $x_0$); in fact, $\alpha^*$ gives a canonical isomorphism of the space of invariant 1-forms on $A$ with the space of closed regular 1-forms on $V$. We then define the 'abelian integral' as

$$^{\mathrm{Ab}}\!\!\int_x^y \omega = {}^{\mathrm{Ab}}\!\!\int_0^{[y-x]} \omega_A = \log_{\omega_A}[y - x] \,.$$

It satisfies (1), (2), and (4) for morphisms of varieties. If one extends it to arbitrary algebraic closed 1-forms as in [Col], then it also satisfies (3) (for a suitable class of functions) and (6). Also this more general integral depends only on the endpoints, not on a path between them. This makes it clear that it differs from Berkovich's integral, which in general *does* depend on the chosen path.

(This does not contradict the uniqueness statement in Theorem 8.1, because the abelian integral is restricted to algebraic objects and morphisms, whereas Berkovich's theorem talks about integration for analytic 1-forms, spaces and maps.)

The abelian integral has some useful applications, which derive from the following easy result. The *rank* of an abelian group $G$ is defined as $\operatorname{rk} G = \dim_{\mathbb{Q}} G \otimes_{\mathbb{Z}} \mathbb{Q}$, or equivalently, the maximal number of elements of $G$ that are linearly independent over $\mathbb{Z}$ ($\operatorname{rk} G := \infty$ if this number is unbounded). **DEF** rank

8.4. **Theorem.** *Let $A$ be an abelian variety over $\mathbb{C}_p$ and let $\Gamma \subset A(\mathbb{C}_p)$ be a subgroup of rank $\operatorname{rk}\Gamma = r < g = \dim A$. Then there are at least $g - r$ linearly independent (over $\mathbb{C}_p$) invariant 1-forms $\omega$ such that $\log_\omega \gamma = 0$ for all $\gamma \in \Gamma$.* **THM** 1-forms killing $\Gamma$

*Proof.* Write $\Omega^1(A)$ for the $\mathbb{C}_p$-vector space of invariant 1-forms on $A$. This space has dimension $g = \dim A$. The map

$$\Omega^1(A) \times A(\mathbb{C}_p) \longrightarrow \mathbb{C}_p, \quad (\omega, x) \longmapsto \log_\omega x$$

is $\mathbb{C}_p$-linear in $\omega$ and $\mathbb{Z}$-linear in $x$. Let $\gamma_1, \ldots, \gamma_r \in \Gamma$ be $\mathbb{Z}$-linearly independent; then for each $\gamma \in \Gamma$ there are integers $m > 0$ and $m_1, \ldots, m_r$ such that

$$m\gamma = m_1\gamma_1 + \ldots + m_r\gamma_r \,. \tag{8.1}$$

Now let $V \subset \Omega^1(A)$ be the subspace given by

$$V = \{\omega \in \Omega^1(A) : \log_\omega \gamma_j = 0 \text{ for } j = 1, \ldots, r\} \,.$$

Since we are imposing $r$ linear conditions, we have $\dim_{\mathbb{C}_p} V \geq g - r$. The relation (8.1) then implies that $\log_\omega \gamma = 0$ for all $\gamma \in \Gamma$. So any $\mathbb{C}_p$-basis of $V$ does what is required. ❑

8.5. **Corollary.** *Let $C$ be a curve of genus $g > 0$ defined over a number field $K$, with Jacobian variety $J$. Let $P_0 \in C(K)$; denote by $i$ the associated embedding $C \to J$ sending $P_0$ to the origin. Let $K_v$ be the completion of $K$ at some $p$-adic place $v$. If $\operatorname{rk} J(K) = r < g$, then there are at least $g - r > 0$ $K_v$-linearly independent 1-forms $\omega$ on $C$ over $K_v$ such that* **COR killing rational points on a curve**

$$^{\mathrm{Ab}}\!\!\int_{P_0}^{P} \omega = 0 \qquad \textit{for all } P \in C(K).$$

*Proof.* We apply Theorem 8.4 and its proof to $\Gamma = J(K) \subset J(K_v) \subset J(\mathbb{C}_v)$. If we restrict to 1-forms defined over $K_v$, the proof goes through as before, working with vector spaces over $K_v$ instead of over $\mathbb{C}_p$. This gives us $\omega_1, \ldots, \omega_{g-r} \in \Omega^1(J/K_v)$ such that $\log_{\omega_j} x = 0$ for all $j$ and all $x \in J(K)$. Pulling back 1-forms induces an isomorphism between $\Omega^1(J/K_v)$ and $\Omega^1(C/K_v)$, and we have for any $j$ and any $P \in C(K)$

$$^{\mathrm{Ab}}\!\!\int_{P_0}^{P} i^*\omega_j = {}^{\mathrm{Ab}}\!\!\int_{0}^{i(P)} \omega_j = \log_{\omega_j} i(P) = 0 \,,$$

so $i^*\omega_1, \ldots, i^*\omega_{g-r}$ do what we want. ❑

We would like to obtain a uniform bound for the number of points in $C(K)$. The following result allows us to reduce to disks and annuli.

8.6. **Lemma.** *Let $C$ and $K_v$ be as above. Then there is a finite collection of analytic maps $\phi_m \colon \mathcal{D}(0,1)^- \to C^{\mathrm{an}}$ and $\psi_n \colon \mathcal{A}(\rho_n, 1)^- \to C^{\mathrm{an}}$ with $\rho_n \in |K_v|^\times$ (where $C^{\mathrm{an}}$ is defined with respect to $K_v \subset \mathbb{C}_p$) defined over $K_v$ such that* **LEMMA partition of $C(/K_v)$**

$$C(K_v) = \bigcup_m \phi_m\big(D(0,1)^- \cap K_v\big) \cup \bigcup_n \psi_n\big(A(\rho_n, 1)^- \cap K_v\big) \,,$$

*and the number of the $\phi_m$ and the $\psi_n$ each are bounded in terms of $g$ and $K_v$ only.*

*Proof.* See [Sto, Prop. 5.3]. The numer of annuli is $\leq 3g - 3$, and the number of disks is in $O(qg)$, where $q$ is the size of the residue field of $K_v$. ❑

Write $D_m = \phi_m(D(0,1)^- \cap K_v)$ and $A_n = \psi_n(A(\rho_n, 1)^- \cap K_v)$. It then suffices to bound $\#\bigl(C(K) \cap D_m\bigr)$ and $\bigl(C(K) \cap A_n\bigr)$ for each $m$ and $n$. Let $V \subset \Omega^1(C/K_v)$ be the space of regular 1-forms that kill $C(K)$ as in Corollary 8.5. Then

$$C(K) \cap D_m \subset \Bigl\{P \in D_m : {}^{\mathrm{Ab}}\!\!\int_{P_0}^{P} \omega = 0\Bigr\}$$

for every $\omega \in V$, and similarly for $A_n$. For a disk $D_m$ we have for $\tau \in K_v$ with $|\tau|_p < 1$

$${}^{\mathrm{Ab}}\!\!\int_{P_0}^{\phi_m(\tau)} \omega = {}^{\mathrm{Ab}}\!\!\int_{P_0}^{\phi_m(0)} \omega + {}^{\mathrm{Ab}}\!\!\int_{\phi_m(0)}^{\phi_m(\tau)} \omega = {}^{\mathrm{Ab}}\!\!\int_{P_0}^{\phi_m(0)} \omega + \int_0^{\tau} \phi_m^* \omega =: h(\tau)$$

(on a disk there is no difference between the two integrals; this comes from the definition of $\log_\omega$ on $U$ by formal integration of power series on the polydisk $D$). Let $\pi$ be a uniformizer of $K_v$ (i.e., $|\pi|_p < 1$ and generates the value group of $K_v$). Then the number of zeros of $h$ in $K_v \cap D(0,1)^- \subset D(0, |\pi|_p)$ is bounded in terms of the number of zeros of $\phi_m^*\omega$ in $D(0,1)^-$ and $|\pi|_p$ (see Exercises). The number of zeros of $\phi_m^*\omega$ in $D(0,1)^-$ is the same as the number of zeros of $\omega$ in $\phi_m(D(0,1)^-)$, which is at most the total number of zeros of $\omega$, which in turn is $2g-2$. This gives us a bound for $\#(C(K) \cap D_m)$ that only depends on $g$ and $K_v$ (via $|\pi|_p$). Since the total number of disks $D_m$ is bounded in terms of $g$ and $K_v$, we get a bound on the total number of $K$-points of $C$ contained in some $D_m$ that only depends on $g$ and $K_v$.

For annuli, the situation is more complicated. There is the following theorem that compares the two integrals on an annulus.

8.7. **Theorem.** *Fix a $K_v$-defined analytic map $\psi\colon \mathcal{A}(\rho, 1)^- \to J^{\mathrm{an}}$. There is a linear form $\alpha\colon \Omega^1(J/K_v) \to K_v$ such that for any $x, y \in A(\rho, 1)^-$ we have* **THM** comparison of integrals on annuli

$$\int_x^y \psi^*\omega - {}^{\mathrm{Ab}}\!\!\int_{\psi(x)}^{\psi(y)} \omega = \alpha(\omega)\bigl(v_p(y) - v_p(x)\bigr)$$

*for all $\omega \in \Omega^1(J/K_v)$.*

*Proof.* See [Sto, Prop. 7.3] or [KRZB, Prop. 3.29]. ❑

If $r \le g-3$, then for each annulus $A_n$ there will be at least one $0 \ne \omega \in V$ such that $\psi_n^*\omega$ is exact and $\alpha(\omega) = 0$ (note that $\alpha$ depends on the annulus), since this imposes two linear conditions. If $\psi_n^*\omega = dh$ is exact, then we can again bound the number of zeros of $h$ in $K_v \cap A(\rho_n, 1)^- \subset A(\rho_n/|\pi|_p, |\pi|_p)$ in terms of $g$ and $K_v$, under some additional assumption (the 'relevant' range of exponents in the Laurent series representation of $h$ includes 0), which can be shown to always hold (this is done in [Sto, Cor. 6.7] for hyperelliptic curves and in [KRZB, Lemma 4.15] in general). This finally leads to the following result.

8.8. **Theorem.** *Fix $d \geq 1$ and $g \geq 3$. There is some $B = B(d,g)$ depending only on $d$ and $g$ such that for any number field $K$ with $[K : \mathbb{Q}] \leq d$ and any curve $C$ of genus $g$ over $K$, with Jacobian $J$ such that* $\operatorname{rk} J(K) \leq g-3$, *we have $\#C(K) \leq B$.*

**THM uniform bound on rational points**

*Proof.* We can assume that $C(K) \neq \emptyset$; let $P_0 \in C(K)$ and use it as the base-point for an embedding $i \colon C \to J$ as in Corollary 8.5. Fix some prime $p$; there are then only finitely many possibilities (up to isomorphism) for the completion $K_v$ at a $p$-adic place $v$. Let $B(d,g)$ be the maximum of the bounds for $C(K)$ obtained by the reasoning above for $g$ and each $K_v$. The claim follows. ❑

For $K = \mathbb{Q}$, one can write down explicit bounds. For hyperelliptic curves (which allow for quite explicit computations) a possible bound is

$$\#C(\mathbb{Q}) \leq 33(g-1) + \begin{cases} 1 & \text{if } r = 0 \text{ (and } g \geq 3), \\ 8rg - 1 & \text{if } 1 \leq r \leq g-3, \end{cases}$$

where $r = \operatorname{rk} J(\mathbb{Q})$, see [Sto, Thm. 9.1] (this works with $\mathbb{Q}_3$). In the general case, one obtains a bound of the shape $O(g^2)$, see [KRZB, Thm. 5.1]. For arbitrary $d$, one still has a bound of the shape $O_d(g^2)$, but the implied constant depends exponentially on $d$.

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