Extending Rational Diophantine Quadruples

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Universität Bayreuth

Salzbug Mathematics Colloquium

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Diophantine m-Tuples

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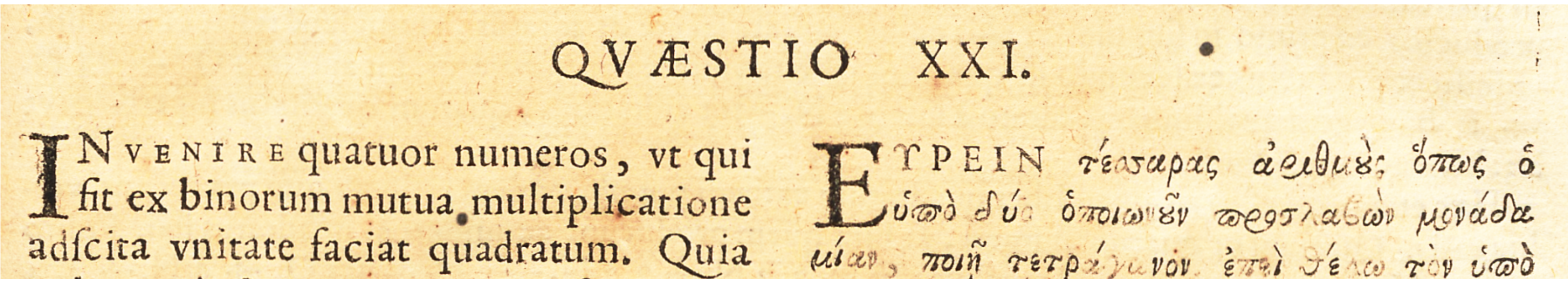
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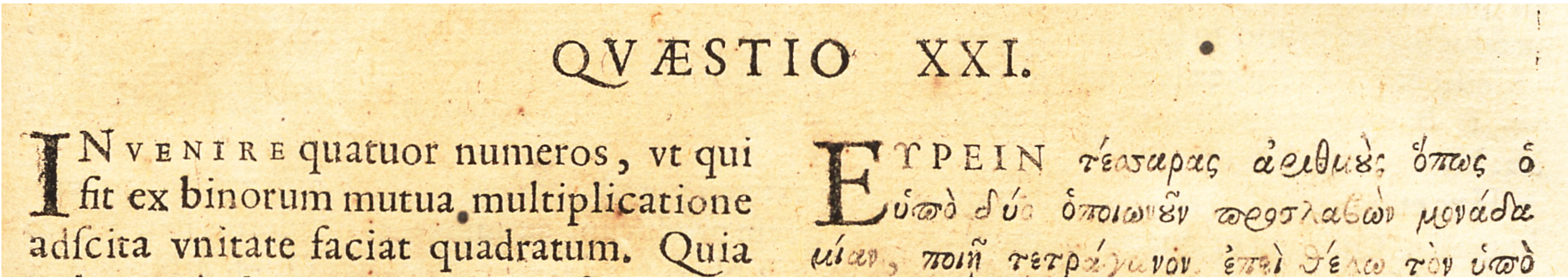
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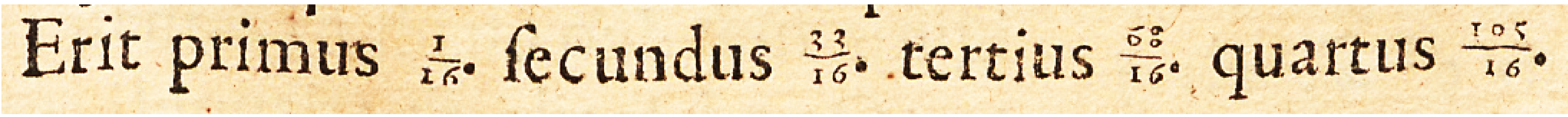
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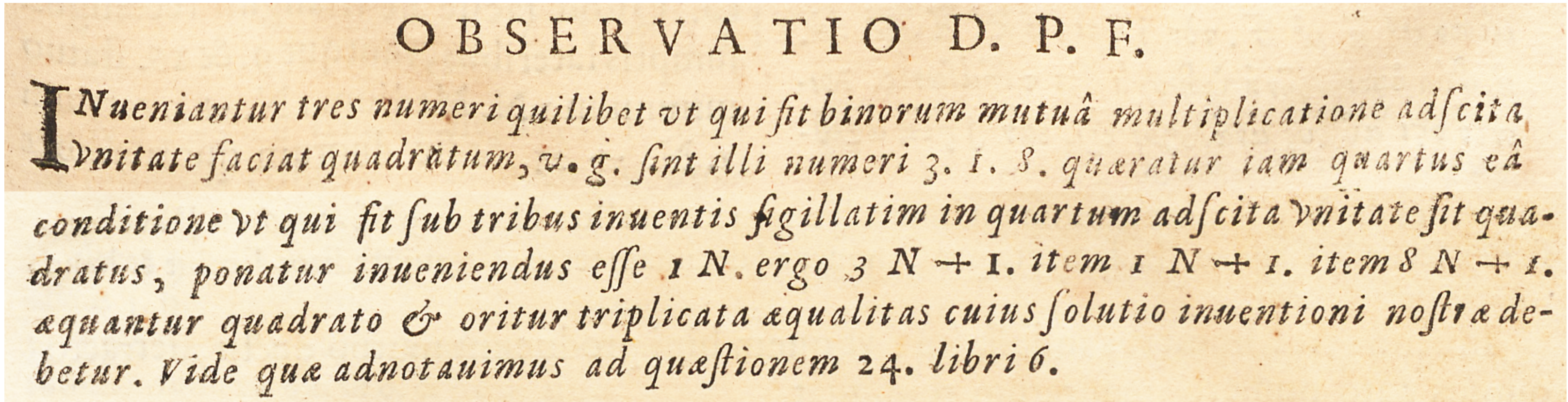
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The gives the Diophantine quadruple $(1, 3, 8, 120)$.

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Euler explicitly asks for (integral) Diophantine quadruples:

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and gives a solution extending every Diophantine pair to two quadruples:

oportet. Nunc igitur ſequentem ſolutionem ſatis ſimplicem exhibeo. Sumtis pro lubitu duobus numeris m et n, vt fiat $mn+1=ll$, quatuor numeri quaeſiti erunt:

I. m; II. n; III. $m+n+2l$; IV. $4l(l+m)(l+n)$,

quibus 6 conditiones praeſcriptae ſequenti modo implentur:

1°. $mn+1=ll$.

2°. $m(m+n+2l)+1=(l+m)^2$.

3°. $n(m+n+2l)+1=(l+n)^2$.

4°. $4m.l(l+m)(l+n)+1=(2ll+2lm-1)^2$.

5°. $4n.l(l+m)(l+n)+1=(2ll+2ln-1)^2$

denique

6°. $4l(m+n+2l)(l+m)(l+n)+1=(4ll+2lm+2ln-1)^2$.

Hic vero plurimum obſeruaſſe iuuabit numerum l tam poſitiue quam negatiue accipi poſſe. Ita ſi ſumatur $m=3$ et $n=8$, vt fiat $mn+1=25$, ideoque $l=\pm 5$, caſus $l=-5$ dabit hos quatuor numeros:

I. 3. II. 8. III. 1 et IV. 120.

Sin autem capiatur $l=+5$, numeri erunt

I. 3. II. 8. III. 21 et IV. 2080.

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Euler then goes a step further:

Problema 5.

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Euler then goes a step further:

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and gives a solution method leading to this example:

Exemplum 1.

Sumamus $m = 1$ et $n = 3$, eritque $l = 2$, vnde quatuor numeri priores erunt $a = 1$; $b = 3$; $c = 8$; $d = 120$ hinc ergo colligimus:

$$p = 132;\ q = 1475;\ r = 4224 \text{ et } s = 2880;$$

ex quibus valoribus deducimus:

$$z = \frac{4 \cdot 4224 + 264 \cdot 2881}{(2879)^2},$$

quae fractio reducitur ad hanc $\frac{777480}{8288641}$, atque hinc decem conditiones praescriptae sequenti modo adimplentur:

1°. $ab + 1 = 2^2$; 2°. $ac + 1 = 3^2$;

3°. $ad + 1 = 11^2$; 4°. $bc + 1 = 5^2$;

5°. $bd + 1 = 19^2$; 6°. $cd + 1 = 31^2$;

7°. $az + 1 = \frac{(3011)^2}{(2879)^2}$; 8°. $bz + 1 = \frac{(3259)^2}{(2879)^2}$;

9°. $cz + 1 = \frac{(3809)^2}{2879^2}$; 10°. $dz + 1 = \frac{10079^2}{2879^2}$.

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There are infinitely many rational Diophantine sextuples;
it is unknown whether rational Diophantine septuples exist or not.

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Consider a given rational Diophantine quadruple (a_1, a_2, a_3, a_4), for example Fermat's quadruple $(1, 3, 8, 120)$.

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Fact.
Euler's method generalizes to give the "regular extensions"

$$a_5 = z_\pm = \frac{(a_1+a_2+a_3+a_4)(a_1a_2a_3a_4+1)+2(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)\pm 2s}{(a_1a_2a_3a_4-1)^2},$$

where $s = \sqrt{(a_1a_2+1)(a_1a_3+1)(a_1a_4+1)(a_2a_3+1)(a_2a_4+1)(a_3a_4+1)}$ (unless $z_\pm \in \{0, a_1, a_2, a_3, a_4\}$).

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Are there more possibilities in our concrete case?

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With $x = u_4$, this gives $x^2 + 119 = 120u_1^2$, $x^2 + 39 = 40u_2^2$, $x^2 + 14 = 15u_3^2$, hence

$$y^2 = 5(x^2 + 119)(x^2 + 39)(x^2 + 14)$$

with $y = 600u_1u_2u_3$.

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Searching further, we do not find any other points.

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So we need to do something else.

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Using a method known as two-cover descent, we can show that, up to possibly a sign change, the x-coordinate of any rational point on C satisfies

$$15(1-\sqrt{-119})(1-\sqrt{-39})\cdot(x^2+14)(x-\sqrt{-119})(x-\sqrt{-39}) = t^2$$

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There is a variant of Chabauty's method that applies in this situation. In our case, its prerequisites are satisfied.

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This finishes the proof.

Further Results

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We have applied this approach to quadruples from the family

$$\left(t-1, t+1, 4t, 4t(4t^2-1)\right)$$

(where $\pm t \neq 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$).

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In this way, we could show that the regular extension is the only one for

$$t = 2 \text{ (see above)},\ 3,\ \tfrac{2}{3},\ \tfrac{3}{2},\ 4,\ \tfrac{3}{4},\ \tfrac{4}{3},\ 5,\ \tfrac{1}{5},\ \tfrac{2}{5},\ \tfrac{3}{5},\ \tfrac{5}{4},\ \tfrac{4}{5}.$$

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(For $t = \frac{3}{5}$, there is a second "illegal" extension besides 0
given by $\frac{12}{5}$, which is already present. Note that $\left(\frac{12}{5}\right)^2 + 1 = \left(\frac{13}{5}\right)^2$.)

Thank You!

Magma Code

```
> P<x> := PolynomialRing(Rationals());
> C := HyperellipticCurve(5*(x^2+119)*(x^2+39)*(x^2+14));
> Sel, mSel := TwoCoverDescent(C);
> A<th> := Domain(mSel);
> assert Sel eq {mSel(x0 - th) : x0 in {1,-1}};
> K := AbsoluteField(ext<Rationals() | x^2 + 119, x^2 + 39>);
> w119 := Roots(x^2 + 119, K)[1,1]; w39 := Roots(x^2 + 39, K)[1,1];
> PK<X> := PolynomialRing(K);
> E := HyperellipticCurve(15*(1-w119)*(1-w39)*(X^2+14)*(X-w119)*(X-w39));
> EE, EtoEE := EllipticCurve(E, E![1, 15*(1-w119)*(1-w39)]);
> assert Invariants(TorsionSubgroup(EE)) eq [2];
> assert #Invariants(TwoSelmerGroup(EE)) eq 3;
> bas := Saturation(ReducedBasis([EtoEE(pt) : pt in Points(E, 10079/2879)]), 7);
> assert #bas eq 3;
> MW := AbelianGroup([2,0,0]);
> MWmap := map<MW -> EE | m :-> &+[s[i]*bas[i] : i in [1..3]] where s := Eltseq(m)>;
> P1 := ProjectiveSpace(Rationals(), 1);
> pi := Expand(Inverse(EtoEE)*map<E -> P1 | [E.1, E.3]>);
> chab := Chabauty(MWmap, pi : IndexBound := 2*3*5*7);
> {pi(MWmap(pt)) : pt in chab};
{ (1 : 1), (10079/2879 : 1) }
```