UNIVERSITÄT BAYREUTH

Minimization and Reduction of Plane Curves

Michael Stoll

Universität Bayreuth

(joint work with Stephan Elsenhans)

Rational Points on Irrational Varieties

Institut Henri Poincaré

June 27, 2019

The Problem

Let K be a field. There is an action of $G(K) = K^\times \times GL(n+1, K)$ on homogenous polynomials of degree d in $n+1$ variables over K via

$$(\lambda, T) \cdot F(x_0, \ldots, x_n) = \lambda \cdot {}^T F(x_0, \ldots, x_n) := \lambda F\big((x_0, \ldots, x_n)T\big) .$$

The Problem

Let K be a field. There is an action of $G(K) = K^\times \times GL(n+1, K)$ on homogenous polynomials of degree d in $n+1$ variables over K via

$$(\lambda, T) \cdot F(x_0, \ldots, x_n) = \lambda \cdot {}^T F(x_0, \ldots, x_n) := \lambda F\big((x_0, \ldots, x_n)T\big) .$$

Problem.

Given $F \in \mathbb{Z}[x_0, \ldots, x_n]$, homogeneous of degree d,
find an integral representative of $G(\mathbb{Q}) \cdot F$ with small coefficients!

The Problem

Let K be a field. There is an action of $G(K) = K^\times \times \mathrm{GL}(n+1, K)$ on homogenous polynomials of degree d in $n+1$ variables over K via

$$(\lambda, T) \cdot F(x_0, \ldots, x_n) = \lambda \cdot {}^T F(x_0, \ldots, x_n) := \lambda F\big((x_0, \ldots, x_n)T\big) .$$

Problem.
Given $F \in \mathbb{Z}[x_0, \ldots, x_n]$, homogeneous of degree d, find an integral representative of $G(\mathbb{Q}) \cdot F$ with small coefficients!

Motivation.

- Small integral equations are nice!

The Problem

Let K be a field. There is an action of $G(K) = K^\times \times GL(n+1, K)$ on homogenous polynomials of degree d in $n+1$ variables over K via

$$(\lambda, T) \cdot F(x_0, \ldots, x_n) = \lambda \cdot {}^T F(x_0, \ldots, x_n) := \lambda F\big((x_0, \ldots, x_n)T\big) .$$

Problem.
Given $F \in \mathbb{Z}[x_0, \ldots, x_n]$, homogeneous of degree d, find an integral representative of $G(\mathbb{Q}) \cdot F$ with small coefficients!

Motivation.

- Small integral equations are nice!
- A reduced minimal equation makes computations easier (e.g., searching for rational points).

The Problem

Let K be a field. There is an action of $G(K) = K^\times \times GL(n+1, K)$ on homogenous polynomials of degree d in $n+1$ variables over K via

$$(\lambda, T) \cdot F(x_0, \ldots, x_n) = \lambda \cdot {}^T F(x_0, \ldots, x_n) := \lambda F\big((x_0, \ldots, x_n)T\big) .$$

Problem.
Given $F \in \mathbb{Z}[x_0, \ldots, x_n]$, homogeneous of degree d,
find an integral representative of $G(\mathbb{Q}) \cdot F$ with small coefficients!

Example.

$$\begin{aligned} F = {} & 19205x^5 + 91317x^4y + 77564x^4z + 167052x^3y^2 + 298432x^3yz + 123837x^3z^2 + 162810x^2y^3 \\ & + 412742x^2y^2z + 358385x^2yz^2 + 96018x^2z^3 + 125740xy^4 + 247220xy^3z + 338880xy^2z^2 \\ & + 177526xyz^3 + 34358xz^4 + 116351y^5 + 19725y^4z + 120660y^3z^2 + 78796y^2z^3 + 25955yz^4 + 3504z^5 \end{aligned}$$

The Problem

Let K be a field. There is an action of $G(K) = K^\times \times \mathrm{GL}(n+1, K)$ on homogenous polynomials of degree d in $n+1$ variables over K via

$$(\lambda, T) \cdot F(x_0, \ldots, x_n) = \lambda \cdot {}^{T}F(x_0, \ldots, x_n) := \lambda F\big((x_0, \ldots, x_n)T\big) .$$

Problem.
Given $F \in \mathbb{Z}[x_0, \ldots, x_n]$, homogeneous of degree d, find an integral representative of $G(\mathbb{Q}) \cdot F$ with small coefficients!

Example.

$$\begin{aligned} F = {} & 19205x^5 + 91317x^4y + 77564x^4z + 167052x^3y^2 + 298432x^3yz + 123837x^3z^2 + 162810x^2y^3 \\ & + 412742x^2y^2z + 358385x^2yz^2 + 96018x^2z^3 + 125740xy^4 + 247220xy^3z + 338880xy^2z^2 \\ & + 177526xyz^3 + 34358xz^4 + 116351y^5 + 19725y^4z + 120660y^3z^2 + 78796y^2z^3 + 25955yz^4 + 3504z^5 \end{aligned}$$

$$\rightsquigarrow \quad \left(\frac{1}{845219547726738091164049}, \begin{pmatrix} -247 & 5681 & -4446 \\ -13338 & 1729 & 3952 \\ -16055 & 3211 & 16055 \end{pmatrix}\right) \cdot F$$

The Problem

Let K be a field. There is an action of $G(K) = K^\times \times GL(n+1, K)$ on homogenous polynomials of degree d in $n+1$ variables over K via

$$(\lambda, T) \cdot F(x_0, \ldots, x_n) = \lambda \cdot {}^T F(x_0, \ldots, x_n) := \lambda F\big((x_0, \ldots, x_n)T\big) .$$

Problem.
Given $F \in \mathbb{Z}[x_0, \ldots, x_n]$, homogeneous of degree d, find an integral representative of $G(\mathbb{Q}) \cdot F$ with small coefficients!

Example.

$$\begin{aligned} F = {} & 19205x^5 + 91317x^4y + 77564x^4z + 167052x^3y^2 + 298432x^3yz + 123837x^3z^2 + 162810x^2y^3 \\ & + 412742x^2y^2z + 358385x^2yz^2 + 96018x^2z^3 + 125740xy^4 + 247220xy^3z + 338880xy^2z^2 \\ & + 177526xyz^3 + 34358xz^4 + 116351y^5 + 19725y^4z + 120660y^3z^2 + 78796y^2z^3 + 25955yz^4 + 3504z^5 \\ \rightsquigarrow \quad & \left(\frac{1}{8452195477267380911640 49}, \begin{pmatrix} -247 & 5681 & -4446 \\ -13338 & 1729 & 3952 \\ -16055 & 3211 & 16055 \end{pmatrix}\right) \cdot F = x^5 - x^2yz^2 + xy^4 - y^5 - z^5 \end{aligned}$$

Two Subproblems

Problem 1 (Minimization).
Remove powers of primes from the invariants of F.
(I.e., improve the reduction mod p.)

Two Subproblems

Problem 1 (Minimization).
Remove powers of primes from the invariants of F.
(I.e., improve the reduction mod p.)

This results in a minimal representative of $G(\mathbb{Q}) \cdot F$,
which still can have large coefficients.

Two Subproblems

Problem 1 (Minimization).
Remove powers of primes from the invariants of F.
(I.e., improve the reduction mod p.)

This results in a minimal representative of $G(\mathbb{Q}) \cdot F$,
which still can have large coefficients.

Necessary: F has some nonzero invariant (i.e., F is not a nullform)

Two Subproblems

Problem 1 (Minimization).
Remove powers of primes from the invariants of F.
(I.e., improve the reduction mod p.)

This results in a minimal representative of $G(\mathbb{Q}) \cdot F$,
which still can have large coefficients.

Necessary: F has some nonzero invariant (i.e., F is not a nullform)

Problem 2 (Reduction).
Find a unimodular transformation $T \in \mathrm{GL}(n+1, \mathbb{Z})$
(this does not change the invariants)
such that ${}^{T}F$ has small coefficients.

Two Subproblems

Problem 1 (Minimization).
Remove powers of primes from the invariants of F.
(I.e., improve the reduction mod p.)

This results in a minimal representative of $G(\mathbb{Q}) \cdot F$,
which still can have large coefficients.

Necessary: F has some nonzero invariant (i.e., F is not a nullform)

Problem 2 (Reduction).
Find a unimodular transformation $T \in GL(n+1, \mathbb{Z})$
(this does not change the invariants)
such that ${}^{T}F$ has small coefficients.

In this talk, we will mainly focus on Problem 1 for ternary forms F.

Minimization at a Prime

We can do minimization for each prime p separately.

Minimization at a Prime

We can do minimization for each prime p separately.

Minimization at p can always be obtained as follows.

$$F \quad \rightsquigarrow \quad F_1 = p^{-e} \cdot {}^{T}F(p^{w_0}x_0, \ldots, p^{w_n}x_n)$$

with T unimodular and $w_0, \ldots, w_n, e \geq 0$ such that $(n+1)e > d(w_0 + \ldots + w_n)$.
We call $w = (w_0, \ldots, w_n)$ a weight vector (cf. Kollár 1997).

Minimization at a Prime

We can do minimization for each prime p separately.

Minimization at p can always be obtained as follows.

$$F \quad \rightsquigarrow \quad F_1 = p^{-e} \cdot {}^{T}F(p^{w_0}x_0, \ldots, p^{w_n}x_n)$$

with T unimodular and $w_0, \ldots, w_n, e \geq 0$ such that $(n+1)e > d(w_0 + \ldots + w_n)$.
We call $w = (w_0, \ldots, w_n)$ a weight vector (cf. Kollár 1997).

Theorem (E&S).
Assume that $n = 2$. If $F \in \mathbb{Z}[x, y, z]_d$ is non-minimal at p,
then F can be (partially) minimized using $w = (0, w_1, w_2)$
with $0 \leq w_1 \leq w_2 \leq d$.

Minimization at a Prime

We can do minimization for each prime p separately.

Minimization at p can always be obtained as follows.

$$F \quad \rightsquigarrow \quad F_1 = p^{-e} \cdot {}^{T}F(p^{w_0}x_0, \ldots, p^{w_n}x_n)$$

with T unimodular and $w_0, \ldots, w_n, e \geq 0$ such that $(n+1)e > d(w_0 + \ldots + w_n)$.
We call $w = (w_0, \ldots, w_n)$ a weight vector (cf. Kollár 1997).

Theorem (E&S).
Assume that $n = 2$. If $F \in \mathbb{Z}[x, y, z]_d$ is non-minimal at p,
then F can be (partially) minimized using $w = (0, w_1, w_2)$
with $0 \leq w_1 \leq w_2 \leq d$.

Minimal complete sets of weight vectors for $n = 2$ (excluding $(0, 0, 0)$).

Minimization at a Prime

We can do minimization for each prime p separately.

Minimization at p can always be obtained as follows.

$$F \quad \rightsquigarrow \quad F_1 = p^{-e} \cdot {}^{T}F(p^{w_0}x_0, \ldots, p^{w_n}x_n)$$

with T unimodular and $w_0, \ldots, w_n, e \geq 0$ such that $(n+1)e > d(w_0 + \ldots + w_n)$.
We call $w = (w_0, \ldots, w_n)$ a weight vector (cf. Kollár 1997).

Theorem (E&S).
Assume that $n = 2$. If $F \in \mathbb{Z}[x, y, z]_d$ is non-minimal at p,
then F can be (partially) minimized using $w = (0, w_1, w_2)$
with $0 \leq w_1 \leq w_2 \leq d$.

Minimal complete sets of weight vectors for $n = 2$ (excluding $(0, 0, 0)$).
$d = 2$: $\quad (0, 0, 1), (0, 1, 1)$

Minimization at a Prime

We can do minimization for each prime p separately.

Minimization at p can always be obtained as follows.

$$F \quad \rightsquigarrow \quad F_1 = p^{-e} \cdot {}^T F(p^{w_0} x_0, \dots, p^{w_n} x_n)$$

with T unimodular and $w_0, \dots, w_n, e \geq 0$ such that $(n+1)e > d(w_0 + \dots + w_n)$.
We call $w = (w_0, \dots, w_n)$ a weight vector (cf. Kollár 1997).

Theorem (E&S).
Assume that $n = 2$. If $F \in \mathbb{Z}[x, y, z]_d$ is non-minimal at p,
then F can be (partially) minimized using $w = (0, w_1, w_2)$
with $0 \leq w_1 \leq w_2 \leq d$.

Minimal complete sets of weight vectors for $n = 2$ (excluding $(0, 0, 0)$).

$d = 2$: $(0, 0, 1), (0, 1, 1)$

$d = 3$: $(0, 0, 1), (0, 1, 1), (0, 1, 2), (0, 2, 3)$

Minimization at a Prime

We can do minimization for each prime p separately.

Minimization at p can always be obtained as follows.

$$F \quad \rightsquigarrow \quad F_1 = p^{-e} \cdot {}^{T}F(p^{w_0}x_0, \ldots, p^{w_n}x_n)$$

with T unimodular and $w_0, \ldots, w_n, e \geq 0$ such that $(n+1)e > d(w_0 + \ldots + w_n)$.
We call $w = (w_0, \ldots, w_n)$ a weight vector (cf. Kollár 1997).

Theorem (E&S).
Assume that $n = 2$. If $F \in \mathbb{Z}[x, y, z]_d$ is non-minimal at p,
then F can be (partially) minimized using $w = (0, w_1, w_2)$
with $0 \leq w_1 \leq w_2 \leq d$.

Minimal complete sets of weight vectors for $n = 2$ (excluding $(0, 0, 0)$).

$d = 2$: $(0, 0, 1), (0, 1, 1)$

$d = 3$: $(0, 0, 1), (0, 1, 1), (0, 1, 2), (0, 2, 3)$

$d = 4$: $(0, 0, 1), (0, 1, 1), (0, 1, 3)$

Minimization at a Prime

We can do minimization for each prime p separately.

Minimization at p can always be obtained as follows.

$$F \quad \rightsquigarrow \quad F_1 = p^{-e} \cdot {}^{T}F(p^{w_0}x_0, \ldots, p^{w_n}x_n)$$

with T unimodular and $w_0, \ldots, w_n, e \geq 0$ such that $(n+1)e > d(w_0 + \ldots + w_n)$.
We call $w = (w_0, \ldots, w_n)$ a weight vector (cf. Kollár 1997).

Theorem (E&S).
Assume that $n = 2$. If $F \in \mathbb{Z}[x, y, z]_d$ is non-minimal at p,
then F can be (partially) minimized using $w = (0, w_1, w_2)$
with $0 \leq w_1 \leq w_2 \leq d$.

Minimal complete sets of weight vectors for $n = 2$ (excluding $(0, 0, 0)$).

$d = 2$: $(0,0,1), (0,1,1)$

$d = 3$: $(0,0,1), (0,1,1), (0,1,2), (0,2,3)$

$d = 4$: $(0,0,1), (0,1,1), (0,1,3)$

$d = 5$: $(0,0,1), (0,1,1), (0,1,2), (0,1,3), (0,2,3), (0,3,4)$

Warmup: Minimization of Binary Forms

For binary forms, $(0, 1)$ is the only relevant nonzero weight vector.

Warmup: Minimization of Binary Forms

For binary forms, $(0, 1)$ is the only relevant nonzero weight vector.
For it to apply with $T = I_2$, the condition for

$$F = a_0 x^d + a_1 x^{d-1} y + \ldots + a_d y^d$$

is $p^e \mid F(x, py)$ with $2e > d$,

Warmup: Minimization of Binary Forms

For binary forms, $(0,1)$ is the only relevant nonzero weight vector.
For it to apply with $T = I_2$, the condition for

$$F = a_0 x^d + a_1 x^{d-1} y + \ldots + a_d y^d$$

is $p^e \mid F(x, py)$ with $2e > d$, i.e.,

$$v_p(a_j) > \frac{d}{2} - j \qquad \text{for } j = 0, \ldots, d.$$

Warmup: Minimization of Binary Forms

For binary forms, $(0, 1)$ is the only relevant nonzero weight vector.
For it to apply with $T = I_2$, the condition for

$$F = a_0 x^d + a_1 x^{d-1} y + \ldots + a_d y^d$$

is $p^e \mid F(x, py)$ with $2e > d$, i.e.,

$$v_p(a_j) > \frac{d}{2} - j \qquad \text{for } j = 0, \ldots, d.$$

This implies that $\bar{F} = F \bmod p$ has a root of multiplicity $> d/2$.

Warmup: Minimization of Binary Forms

For binary forms, $(0, 1)$ is the only relevant nonzero weight vector.
For it to apply with $T = I_2$, the condition for

$$F = a_0 x^d + a_1 x^{d-1} y + \ldots + a_d y^d$$

is $p^e \mid F(x, py)$ with $2e > d$, i.e.,

$$v_p(a_j) > \frac{d}{2} - j \qquad \text{for } j = 0, \ldots, d.$$

This implies that $\bar{F} = F \bmod p$ has a root of multiplicity $> d/2$.

Algorithm.

1. **if** no root of $\bar{F}$ has multiplicity $> d/2$ **then return** F
2.
3.
4.

Warmup: Minimization of Binary Forms

For binary forms, $(0, 1)$ is the only relevant nonzero weight vector.
For it to apply with $T = I_2$, the condition for

$$F = a_0 x^d + a_1 x^{d-1} y + \ldots + a_d y^d$$

is $p^e \mid F(x, py)$ with $2e > d$, i.e.,

$$v_p(a_j) > \frac{d}{2} - j \qquad \text{for } j = 0, \ldots, d.$$

This implies that $\bar{F} = F \bmod p$ has a root of multiplicity $> d/2$.

Algorithm.

1. **if** no root of $\bar{F}$ has multiplicity $> d/2$ **then return** F
2. Find $T \in \mathrm{GL}(2, \mathbb{Z})$ such that the high-mult. root of $\overline{{}^T F}$ is at $(1 : 0)$.
3.
4.

Warmup: Minimization of Binary Forms

For binary forms, $(0, 1)$ is the only relevant nonzero weight vector. For it to apply with $T = I_2$, the condition for

$$F = a_0 x^d + a_1 x^{d-1} y + \ldots + a_d y^d$$

is $p^e \mid F(x, py)$ with $2e > d$, i.e.,

$$v_p(a_j) > \frac{d}{2} - j \qquad \text{for } j = 0, \ldots, d.$$

This implies that $\bar{F} = F \bmod p$ has a root of multiplicity $> d/2$.

Algorithm.

1. **if** no root of $\bar{F}$ has multiplicity $> d/2$ **then return** F
2. Find $T \in \mathrm{GL}(2, \mathbb{Z})$ such that the high-mult. root of $\overline{{}^T F}$ is at $(1 : 0)$.
3. $e := v_p({}^T F(x, py))$
4.

Warmup: Minimization of Binary Forms

For binary forms, $(0,1)$ is the only relevant nonzero weight vector.
For it to apply with $T = I_2$, the condition for

$$F = a_0 x^d + a_1 x^{d-1} y + \ldots + a_d y^d$$

is $p^e \mid F(x, py)$ with $2e > d$, i.e.,

$$v_p(a_j) > \frac{d}{2} - j \qquad \text{for } j = 0, \ldots, d.$$

This implies that $\bar{F} = F \bmod p$ has a root of multiplicity $> d/2$.

Algorithm.

1. **if** no root of $\bar{F}$ has multiplicity $> d/2$ **then return** F
2. Find $T \in \mathrm{GL}(2, \mathbb{Z})$ such that the high-mult. root of $\overline{{}^T F}$ is at $(1:0)$.
3. $e := v_p({}^T F(x, py))$
4. **if** $2e > d$ **then** $F := p^{-e} \cdot {}^T F(x, py)$; **goto** 1. **else return** F

Warmup: Minimization of Binary Forms

For binary forms, $(0, 1)$ is the only relevant nonzero weight vector.
For it to apply with $T = I_2$, the condition for

$$F = a_0 x^d + a_1 x^{d-1} y + \ldots + a_d y^d$$

is $p^e \mid F(x, py)$ with $2e > d$, i.e.,

$$v_p(a_j) > \frac{d}{2} - j \qquad \text{for } j = 0, \ldots, d.$$

This implies that $\bar{F} = F \bmod p$ has a root of multiplicity $> d/2$.

Algorithm.

1. **if** no root of $\bar{F}$ has multiplicity $> d/2$ **then return** F
2. Find $T \in GL(2, \mathbb{Z})$ such that the high-mult. root of $\overline{{}^T F}$ is at $(1 : 0)$.
3. $e := v_p({}^T F(x, py))$
4. **if** $2e > d$ **then** $F := p^{-e} \cdot {}^T F(x, py)$; **goto** 1. **else return** F

(For reduction of binary forms, see Cremona&Stoll 2003.)

Geometric Conditions for Ternary Forms

We need an efficient way to find a suitable unimodular T.

Geometric Conditions for Ternary Forms

We need an efficient way to find a suitable unimodular T.

If $w = (0, 0, 1)$ applies, then $\bar{F}$ splits off a high-multiplicity linear factor.
If $w = (0, 1, 1)$ applies, then $\{\bar{F} = 0\}$ has a point of high multiplicity.

Geometric Conditions for Ternary Forms

We need an efficient way to find a suitable unimodular T.

If $w = (0, 0, 1)$ applies, then $\bar{F}$ splits off a high-multiplicity linear factor.
If $w = (0, 1, 1)$ applies, then $\{\bar{F} = 0\}$ has a point of high multiplicity.

We denote by X the projective plane curve over $\mathbb{F}_p$ given by $\bar{F} = 0$.

Geometric Conditions for Ternary Forms

We need an efficient way to find a suitable unimodular T.

If $w = (0, 0, 1)$ applies, then $\bar{F}$ splits off a high-multiplicity linear factor.
If $w = (0, 1, 1)$ applies, then $\{\bar{F} = 0\}$ has a point of high multiplicity.

We denote by X the projective plane curve over $\mathbb{F}_p$ given by $\bar{F} = 0$.

Theorem (E&S).
If $F \in \mathbb{Z}[x, y, z]$ is not minimal at p, then we have one of the following.

Geometric Conditions for Ternary Forms

We need an efficient way to find a suitable unimodular T.

If $w = (0, 0, 1)$ applies, then $\bar{F}$ splits off a high-multiplicity linear factor.
If $w = (0, 1, 1)$ applies, then $\{\bar{F} = 0\}$ has a point of high multiplicity.

We denote by X the projective plane curve over $\mathbb{F}_p$ given by $\bar{F} = 0$.

Theorem (E&S).
If $F \in \mathbb{Z}[x, y, z]$ is not minimal at p, then we have one of the following.
(1) X contains a line L with multiplicity $> d/3$;

Geometric Conditions for Ternary Forms

We need an efficient way to find a suitable unimodular T.

If $w = (0, 0, 1)$ applies, then $\bar{F}$ splits off a high-multiplicity linear factor.
If $w = (0, 1, 1)$ applies, then $\{\bar{F} = 0\}$ has a point of high multiplicity.

We denote by X the projective plane curve over $\mathbb{F}_p$ given by $\bar{F} = 0$.

Theorem (E&S).
If $F \in \mathbb{Z}[x, y, z]$ is not minimal at p, then we have one of the following.
(1) X contains a line L with multiplicity $> d/3$;
(2) X contains a line L with multiplicity $0 < m \leq d/3$
and there is a point $P \in L$ with multiplicity $> (d - m)/2$ on X;

Geometric Conditions for Ternary Forms

We need an efficient way to find a suitable unimodular T.

If $w = (0, 0, 1)$ applies, then $\bar{F}$ splits off a high-multiplicity linear factor.
If $w = (0, 1, 1)$ applies, then $\{\bar{F} = 0\}$ has a point of high multiplicity.

We denote by X the projective plane curve over $\mathbb{F}_p$ given by $\bar{F} = 0$.

Theorem (E&S).
If $F \in \mathbb{Z}[x, y, z]$ is not minimal at p, then we have one of the following.
(1) X contains a line L with multiplicity $> d/3$;
(2) X contains a line L with multiplicity $0 < m \leq d/3$
and there is a point $P \in L$ with multiplicity $> (d - m)/2$ on X;
(3) X has a point P of multiplicity $> d/2$, which is not on a line in X.

Geometric Conditions for Ternary Forms

We need an efficient way to find a suitable unimodular T.

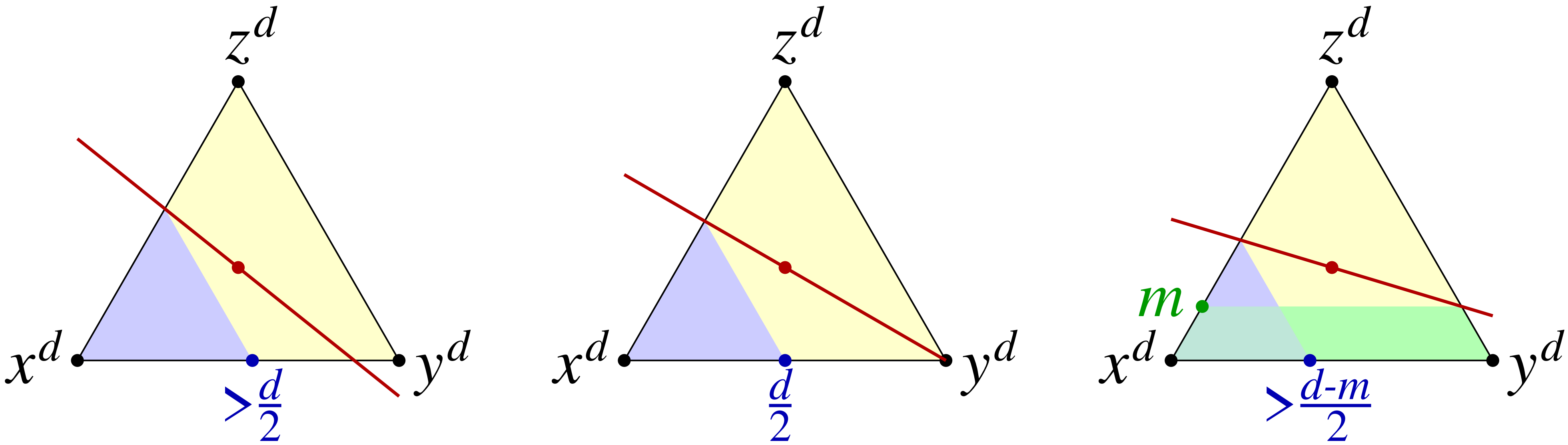
If $w = (0, 0, 1)$ applies, then $\bar{F}$ splits off a high-multiplicity linear factor.
If $w = (0, 1, 1)$ applies, then $\{\bar{F} = 0\}$ has a point of high multiplicity.

We denote by X the projective plane curve over $\mathbb{F}_p$ given by $\bar{F} = 0$.

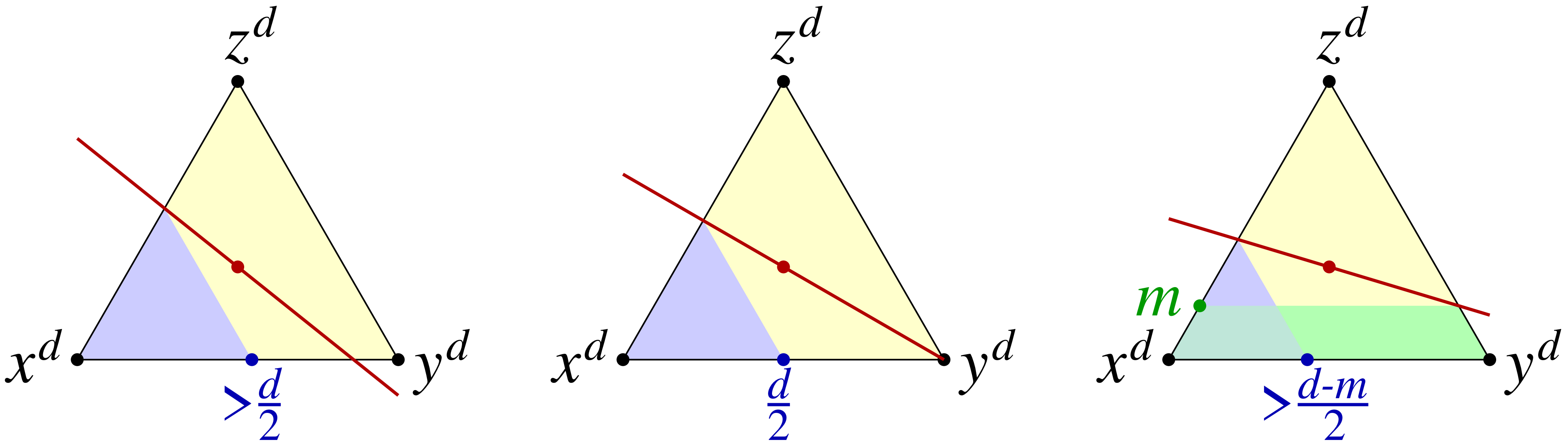
Theorem (E&S).
If $F \in \mathbb{Z}[x, y, z]$ is not minimal at p, then we have one of the following.
(1) X contains a line L with multiplicity $> d/3$;
(2) X contains a line L with multiplicity $0 < m \leq d/3$
and there is a point $P \in L$ with multiplicity $> (d - m)/2$ on X;
(3) X has a point P of multiplicity $> d/2$, which is not on a line in X.

Moving L to $z = 0$ and applying $w = (0, 0, 1)$ in cases (1), (2), or
moving P to $(1 : 0 : 0)$ and applying $w = (0, 1, 1)$ in case (3)
brings us one step/two steps 'closer' to a minimal representative.

Sketch of Proof



Sketch of Proof



Condition for instability:
After a coordinate change, all coefficients on or below the red line vanish, for some choice of red line.

Algorithm for Minimization at p

Let $m(d)$ be a bound on $w_1 + w_2$ over the weight vectors for degree d.
This is the maximum number of steps to a successful partial minimization.

Algorithm for Minimization at p

Let $m(d)$ be a bound on $w_1 + w_2$ over the weight vectors for degree d.
This is the maximum number of steps to a successful partial minimization.

1. Determine a list of pairs (T, w) for F as in the theorem.

2.

2a.

2b.

2c.

2d.

3.

Algorithm for Minimization at p

Let $m(d)$ be a bound on $w_1 + w_2$ over the weight vectors for degree d.
This is the maximum number of steps to a successful partial minimization.

1. Determine a list of pairs (T, w) for F as in the theorem.
2. **for each** pair (T, w) **do**

2a.

2b.

2c.

2d.

3.

Algorithm for Minimization at p

Let $m(d)$ be a bound on $w_1 + w_2$ over the weight vectors for degree d.
This is the maximum number of steps to a successful partial minimization.

1. Determine a list of pairs (T, w) for F as in the theorem.
2. **for each** pair (T, w) **do**

2a. $\tilde{F} := {}^{T}F(x, p^{w_1}y, p^{w_2}z)$; $e := v_p(\tilde{F})$; $F_1 := p^{-e}\tilde{F}$

2b.

2c.

2d.

3.

Algorithm for Minimization at p

Let $m(d)$ be a bound on $w_1 + w_2$ over the weight vectors for degree d.
This is the maximum number of steps to a successful partial minimization.

1. Determine a list of pairs (T, w) for F as in the theorem.
2. **for each** pair (T, w) **do**

2a. $\tilde{F} := {}^{T}F(x, p^{w_1}y, p^{w_2}z)$; $e := v_p(\tilde{F})$; $F_1 := p^{-e}\tilde{F}$

2b. **if** F_1 is a partial minimization of F **then return** F_1

2c.

2d.

3.

Algorithm for Minimization at p

Let $m(d)$ be a bound on $w_1 + w_2$ over the weight vectors for degree d.
This is the maximum number of steps to a successful partial minimization.

1. Determine a list of pairs (T, w) for F as in the theorem.
2. **for each** pair (T, w) **do**

2a. $\tilde{F} := {}^{T}F(x, p^{w_1}y, p^{w_2}z)$; $e := v_p(\tilde{F})$; $F_1 := p^{-e}\tilde{F}$

2b. **if** F_1 is a partial minimization of F **then return** F_1

2c. **if** steps so far $> m(d)$ **then** terminate this branch

2d.

3.

Algorithm for Minimization at p

Let $m(d)$ be a bound on $w_1 + w_2$ over the weight vectors for degree d.
This is the maximum number of steps to a successful partial minimization.

1. Determine a list of pairs (T, w) for F as in the theorem.
2. **for each** pair (T, w) **do**

2a. $\tilde{F} := {}^{T}F(x, p^{w_1}y, p^{w_2}z)$; $e := v_p(\tilde{F})$; $F_1 := p^{-e}\tilde{F}$

2b. **if** F_1 is a partial minimization of F **then return** F_1

2c. **if** steps so far $> m(d)$ **then** terminate this branch

2d. apply the procedure recursively to F_1

3.

Algorithm for Minimization at p

Let $m(d)$ be a bound on $w_1 + w_2$ over the weight vectors for degree d.
This is the maximum number of steps to a successful partial minimization.

1. Determine a list of pairs (T, w) for F as in the theorem.
2. **for each** pair (T, w) **do**

2a. $\tilde{F} := {}^T F(x, p^{w_1} y, p^{w_2} z)$; $e := v_p(\tilde{F})$; $F_1 := p^{-e} \tilde{F}$

2b. **if** F_1 is a partial minimization of F **then return** F_1

2c. **if** steps so far $> m(d)$ **then** terminate this branch

2d. apply the procedure recursively to F_1

3. **return** F // *if we get here, no success*

Algorithm for Minimization at p

Let $m(d)$ be a bound on $w_1 + w_2$ over the weight vectors for degree d.
This is the maximum number of steps to a successful partial minimization.

1. Determine a list of pairs (T, w) for F as in the theorem.
2. **for each** pair (T, w) **do**

2a. $\tilde{F} := {}^{T}F(x, p^{w_1}y, p^{w_2}z)$; $e := v_p(\tilde{F})$; $F_1 := p^{-e}\tilde{F}$

2b. **if** F_1 is a partial minimization of F **then return** F_1

2c. **if** steps so far $> m(d)$ **then** terminate this branch

2d. apply the procedure recursively to F_1

3. **return** F // *if we get here, no success*

We keep track of the accumulated transformation
and of the number of minimization steps made.

Algorithm for Minimization at p

Let $m(d)$ be a bound on $w_1 + w_2$ over the weight vectors for degree d.
This is the maximum number of steps to a successful partial minimization.

1. Determine a list of pairs (T, w) for F as in the theorem.
2. **for each** pair (T, w) **do**

2a. $\tilde{F} := {}^{T}F(x, p^{w_1}y, p^{w_2}z)$; $e := v_p(\tilde{F})$; $F_1 := p^{-e}\tilde{F}$

2b. **if** F_1 is a partial minimization of F **then return** F_1

2c. **if** steps so far $> m(d)$ **then** terminate this branch

2d. apply the procedure recursively to F_1

3. **return** F // *if we get here, no success*

We keep track of the accumulated transformation
and of the number of minimization steps made.

We repeat the procedure above until no further minimization is possible.
This is quite fast in practice.

Determining the Relevant Primes

We need to find the primes p at which F is not minimal.
This is a global computation and tends to be the main bottleneck.

Determining the Relevant Primes

We need to find the primes p at which F is not minimal.
This is a global computation and tends to be the main bottleneck.

We can translate the conditions in the theorem
into systems of polynomial equations and compute Gröbner Bases over $\mathbb{Z}$.
Each p has to divide the integer in one of the GBs (if it exists).

Determining the Relevant Primes

We need to find the primes p at which F is not minimal.
This is a global computation and tends to be the main bottleneck.

We can translate the conditions in the theorem
into systems of polynomial equations and compute Gröbner Bases over $\mathbb{Z}$.
Each p has to divide the integer in one of the GBs (if it exists).

This is still slow. We can speed it up by noting that
all invariants of F must be divisible by p.
So we compute a few invariants and add their gcd to the ideal generators.

Determining the Relevant Primes

We need to find the primes p at which F is not minimal.
This is a global computation and tends to be the main bottleneck.

We can translate the conditions in the theorem
into systems of polynomial equations and compute Gröbner Bases over $\mathbb{Z}$.
Each p has to divide the integer in one of the GBs (if it exists).

This is still slow. We can speed it up by noting that
all invariants of F must be divisible by p.
So we compute a few invariants and add their gcd to the ideal generators.

This results in an implementation that usually runs in reasonable time.

Reduction

The paper

M. Stoll, *Reduction theory of point clusters in projective space*,
Groups Geom. Dyn. **5**, 553–565 (2011)

shows how to find a unimodular transformation
that moves a bunch of points in $\mathbb{P}^n$ to a nice position.

Reduction

The paper

M. Stoll, *Reduction theory of point clusters in projective space*,
Groups Geom. Dyn. **5**, 553–565 (2011)

shows how to find a unimodular transformation
that moves a bunch of points in $\mathbb{P}^{n\dagger}$ to a nice position.

†an effective zero-cycle defined over $\mathbb{R}$

Reduction

The paper

M. Stoll, *Reduction theory of point clusters in projective space*,
Groups Geom. Dyn. **5**, 553–565 (2011)

shows how to find a unimodular transformation
that moves a bunch of points in $\mathbb{P}^{n\dagger}$ to a nice position.

We can apply this to a cluster of points that is covariantly associated to F,
e.g., the cluster of inflection points of $\{F = 0\}$.

†an effective zero-cycle defined over $\mathbb{R}$

Reduction

The paper

M. Stoll, *Reduction theory of point clusters in projective space*,
Groups Geom. Dyn. **5**, 553–565 (2011)

shows how to find a unimodular transformation
that moves a bunch of points in $\mathbb{P}^{n\dagger}$ to a nice position.

We can apply this to a cluster of points that is covariantly associated to F,
e.g., the cluster of inflection points of $\{F = 0\}$.

In practice, a simple ad-hoc reduction works very well:
try 'small' transformations as long as they make the coefficients smaller.

† an effective zero-cycle defined over $\mathbb{R}$

Reduction

The paper

M. Stoll, *Reduction theory of point clusters in projective space*,
Groups Geom. Dyn. **5**, 553–565 (2011)

shows how to find a unimodular transformation
that moves a bunch of points in $\mathbb{P}^n$[†] to a nice position.

We can apply this to a cluster of points that is covariantly associated to F,
e.g., the cluster of inflection points of $\{F = 0\}$.

In practice, a simple ad-hoc reduction works very well:
try 'small' transformations as long as they make the coefficients smaller.

It is also a good idea to apply this between minimization
at different primes, to keep coefficient growth in check.

[†]an effective zero-cycle defined over $\mathbb{R}$

Examples

Here are two plane quartics from a famous paper.

$$F_1 = x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + 18xyz^2 + 9y^2z^2 - 9z^4$$

$$F_2 = -3x^4 - 6x^3z + 6x^2y^2 - 6x^2yz + 15x^2z^2 - 4xy^3 - 6xyz^2 - 4xz^3 + 6y^2z^2 - 6yz^3$$

Examples

Here are two plane quartics from a famous paper.

$$F_1 = x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + 18xyz^2 + 9y^2z^2 - 9z^4$$

$$F_2 = -3x^4 - 6x^3z + 6x^2y^2 - 6x^2yz + 15x^2z^2 - 4xy^3 - 6xyz^2 - 4xz^3 + 6y^2z^2 - 6yz^3$$

Both turn out to be non-minimal at 2; minimal models are given by

$$\tilde{F}_1 = -x^4 - 5x^3y + 4x^3z + 6x^2y^2 - 3x^2z^2 - 2xy^3 - 2xz^3 + 4y^3z - 6y^2z^2 + 4yz^3$$

$$\begin{aligned}\tilde{F}_2 = -x^3y - 3x^2yz + 3x^2z^2 + 3xy^3 - 6xy^2z - 3xyz^2 \\ - 4y^4 - 15y^3z + 21y^2z^2 - 15yz^3 - 3z^4\end{aligned}$$

Examples

Here are two plane quartics from a famous paper.

$$F_1 = x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + 18xyz^2 + 9y^2z^2 - 9z^4$$

$$F_2 = -3x^4 - 6x^3z + 6x^2y^2 - 6x^2yz + 15x^2z^2 - 4xy^3 - 6xyz^2 - 4xz^3 + 6y^2z^2 - 6yz^3$$

Both turn out to be non-minimal at 2; minimal models are given by

$$\tilde{F}_1 = -x^4 - 5x^3y + 4x^3z + 6x^2y^2 - 3x^2z^2 - 2xy^3 - 2xz^3 + 4y^3z - 6y^2z^2 + 4yz^3$$

$$\begin{aligned}\tilde{F}_2 = -x^3y - 3x^2yz + 3x^2z^2 + 3xy^3 - 6xy^2z - 3xyz^2 \\ - 4y^4 - 15y^3z + 21y^2z^2 - 15yz^3 - 3z^4\end{aligned}$$

(Minimization and reduction of plane quartics
was already implemented by SE in Magma.)

Example

Here is a plane sextic from a future famous paper.

$$
\begin{aligned}
F = {} & 5x^6 - 50x^5y + 206x^4y^2 - 408x^3y^3 + 321x^2y^4 + 10xy^5 - 100y^6 \\
& + 9x^4z^2 - 60x^3yz^2 + 80x^2y^2z^2 + 48xy^3z^2 + 15y^4z^2 \\
& + 3x^2z^4 - 10xyz^4 + 6y^2z^4 - z^6
\end{aligned}
$$

Example

Here is a plane sextic from a future famous paper.

$$\begin{aligned} F = 5x^6 - 50x^5y + 206x^4y^2 - 408x^3y^3 + 321x^2y^4 + 10xy^5 - 100y^6 \\ + 9x^4z^2 - 60x^3yz^2 + 80x^2y^2z^2 + 48xy^3z^2 + 15y^4z^2 \\ + 3x^2z^4 - 10xyz^4 + 6y^2z^4 - z^6 \end{aligned}$$

This is the unique plane sextic model with four simple double points of a certain modular curve $X(b5, ns7)$ of genus 6.

Example

Here is a plane sextic from a future famous paper.

$$\begin{aligned}F = 5x^6 - 50x^5y + 206x^4y^2 - 408x^3y^3 + 321x^2y^4 + 10xy^5 - 100y^6\\ + 9x^4z^2 - 60x^3yz^2 + 80x^2y^2z^2 + 48xy^3z^2 + 15y^4z^2\\ + 3x^2z^4 - 10xyz^4 + 6y^2z^4 - z^6\end{aligned}$$

This is the unique plane sextic model with four simple double points of a certain modular curve $X(b5, ns7)$ of genus 6.

It is non-minimal at 2; our algorithm produces the following model.

$$\begin{aligned}\tilde{F} = -x^6 - 2x^5y + 2x^5z + 23x^4yz - 5x^3y^3 - x^3y^2z + x^3yz^2 + 5x^3z^3 - x^2y^4\\ - 8x^2y^3z + 17x^2y^2z^2 - 8x^2yz^3 - x^2z^4 + 3xy^5 - 7xy^4z + 10xy^3z^2\\ - 10xy^2z^3 + 7xyz^4 - 3xz^5 + y^6 - 3y^5z + 3y^4z^2 - 6y^3z^3 + 3y^2z^4 - 3yz^5 + z^6\end{aligned}$$

Live Demonstration

```
> pol := 5*x^6 - 50*x^5*y + 206*x^4*y^2 - 408*x^3*y^3 + 321*x^2*y^4
         + 10*x*y^5 - 100*y^6 + 9*x^4*z^2 - 60*x^3*y*z^2
         + 80*x^2*y^2*z^2 + 48*x*y^3*z^2 + 15*y^4*z^2 + 3*x^2*z^4
         - 10*x*y*z^4 + 6*y^2*z^4 - z^6;
> MinRedTernaryForm(pol);
-x^6 - 2*x^5*y + 2*x^5*z + 23*x^4*y*z - 5*x^3*y^3 - x^3*y^2*z +
    x^3*y*z^2 + 5*x^3*z^3 - x^2*y^4 - 8*x^2*y^3*z + 17*x^2*y^2*z^2 -
    8*x^2*y*z^3 - x^2*z^4 + 3*x*y^5 - 7*x*y^4*z + 10*x*y^3*z^2 -
    10*x*y^2*z^3 + 7*x*y*z^4 - 3*x*z^5 + y^6 - 3*y^5*z + 3*y^4*z^2 -
    6*y^3*z^3 + 3*y^2*z^4 - 3*y*z^5 + z^6

[ 1 -1  1]
[ 1  0  0]
[ 0  1  1]
16
```

Thank You!

Smaller Even Model

$$F = 5x^6 - 50x^5y + 206x^4y^2 - 408x^3y^3 + 321x^2y^4 + 10xy^5 - 100y^6$$
$$+ 9x^4z^2 - 60x^3yz^2 + 80x^2y^2z^2 + 48xy^3z^2 + 15y^4z^2$$
$$+ 3x^2z^4 - 10xyz^4 + 6y^2z^4 - z^6$$

The following is a model with smaller coefficients that preserves the involution $(x, y, z) \mapsto (x, y, -z)$.

$$F(x + 2y, y, z) = 5x^6 + 10x^5y + 6x^4y^2 + 40x^3y^3 + 17x^2y^4 - 50xy^5 - 44y^6$$
$$+ 9x^4z^2 + 12x^3yz^2 - 64x^2y^2z^2 - 64xy^3z^2 + 95y^4z^2$$
$$+ 3x^2z^4 + 2xyz^4 - 2y^2z^4 - z^6$$