UNIVERSITÄT BAYREUTH

Uniform bounds on the number of rational points on curves of low Mordell-Weil rank

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$\mathfrak{p}$-adic Methods in Number Theory

Berkeley

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Motivation (1)

Theorem. (‘Mordell’s Conjecture’, Faltings 1983)
Let C be a (‘nice’) curve of genus $g \geq 2$ over a number field K.
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Theorem. (Caporaso-Harris-Mazur, Pacelli 1997)
The weak Lang Conjecture
(rational points on varieties of general type are not Zariski dense)
implies a positive answer.

Motivation (2)

A geometric variant:

Theorem. ('Mordell-Lang Conjecture', Faltings 1994)
Let C be a curve of genus $g \geq 2$ over $\mathbb{C}$,
with an embedding $i \colon C \to J$ into its Jacobian.
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Is there a uniform bound $N'(g, r)$ (depending only on g and r) for $\#i^{-1}(\Gamma)$?

Theorem.
The Zilber-Pink Conjecture for families of abelian varieties
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Originally for hyperelliptic curves and p odd;

generalized by Katz, Rabinoff & Zureick-Brown (preprint, 2015).

See the next talk, by David Zureick-Brown.

Bound for the Number of Rational Points

Taking $K = \mathbb{Q}_3$, C defined over $\mathbb{Q}$ and $\Gamma = J(\mathbb{Q})$, we obtain:

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Then for every hyperelliptic curve C of genus g over $\mathbb{Q}$
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More generally, there is a bound $N(d, g, r)$ (for $r \leq g - 3$) such that
for all number fields K with $[K : \mathbb{Q}] \leq d$ and
all curves of genus g over K with $J(K)$ of rank r, we have

$$\#C(K) \leq N(d, g, r) = O_d(g^2)\,.$$

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Let C be a curve of genus g over $\mathbb{Q}_p$ of good reduction with $p > 2g$, with (minimal) proper regular model $\mathcal{C}$ over $\mathbb{Z}_p$ (with special fiber $\mathcal{C}_s$).
Let $i\colon C \to J$ be an embedding given by a base-point $P_0 \in C(\mathbb{Q}_p)$ and let $\Gamma \subset J(\mathbb{Q}_p)$ be a subgroup of rank $r \leq g - 1$. Then

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$$\#i^{-1}(\Gamma) \le \#\mathcal{C}_s^{\mathrm{smooth}}(\mathbb{F}_p) + 2r + \left\lfloor \frac{2r}{p-2} \right\rfloor .$$

Problem: $\#\mathcal{C}_s^{\mathrm{smooth}}(\mathbb{F}_p)$ cannot be bounded in terms of g and p!

Sketch of Proof (1)

We have canonical isomorphisms $\Omega^1_{C/\mathbb{Q}_p} \cong \Omega^1_{J/\mathbb{Q}_p} \cong T_0 J(\mathbb{Q}_p)^*$
and the p-adic abelian logarithm $\log_J\colon J(\mathbb{Q}_p) \to T_0 J(\mathbb{Q}_p)$.

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We obtain a pairing $\Omega^1_{C/\mathbb{Q}_p} \times J(\mathbb{Q}_p) \to \mathbb{Q}_p, \quad (\omega, P) \mapsto \langle \omega, P \rangle$
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This pairing is the abelian integral (compare Dick Gross's lecture):
For $P_0, P \in C(\mathbb{Q}_p)$, we have

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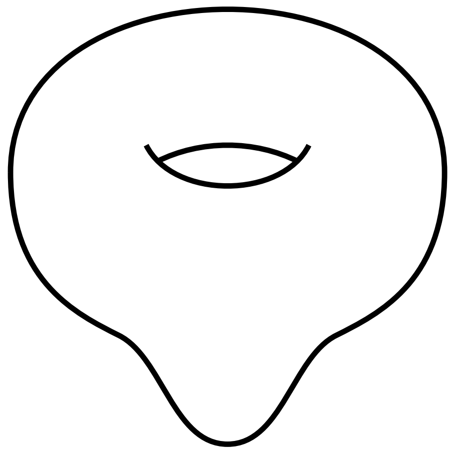
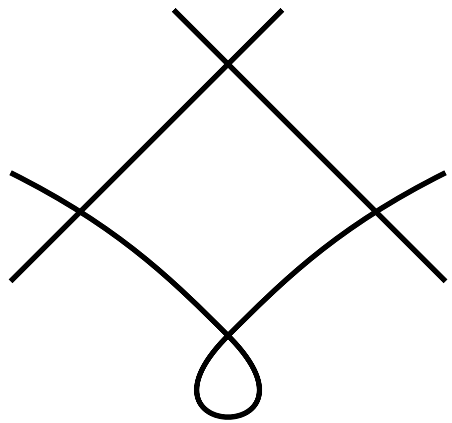
Note that $i^{-1}(\Gamma) \subset \bigl\{P \in C(\mathbb{Q}_p) : \forall \omega \in V_\Gamma \colon \langle \omega, i(P) \rangle = 0\bigr\}$.

A Picture

generic fiber of $\mathcal{C} = \mathrm{C}$

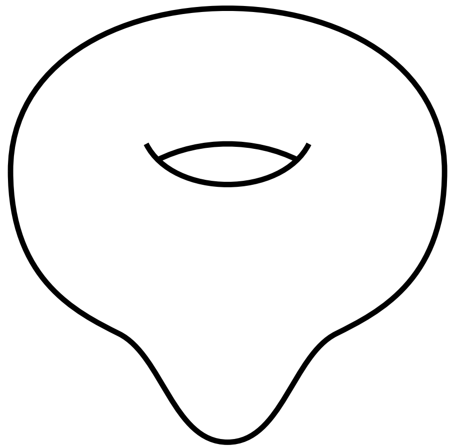


special fiber of $\mathcal{C}$



A Picture

generic fiber of $\mathcal{C} = C$



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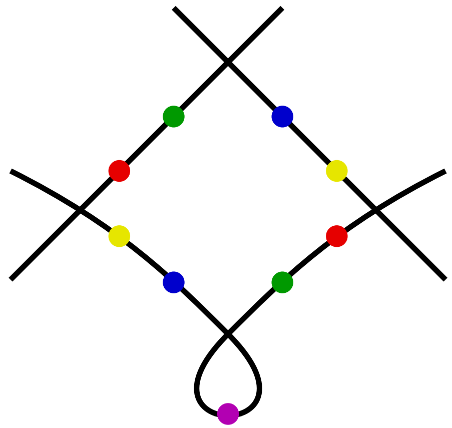

$\mathcal{C}_{\mathsf{s}}^{\mathsf{smooth}}(\mathbb{F}_p)$

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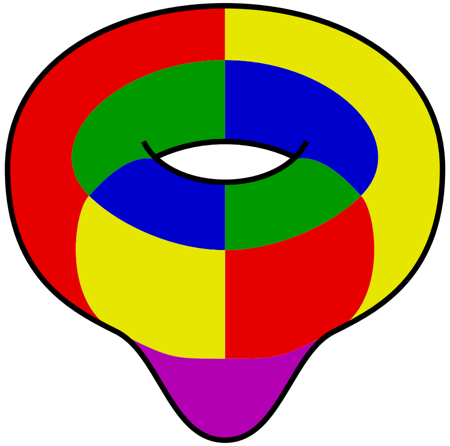
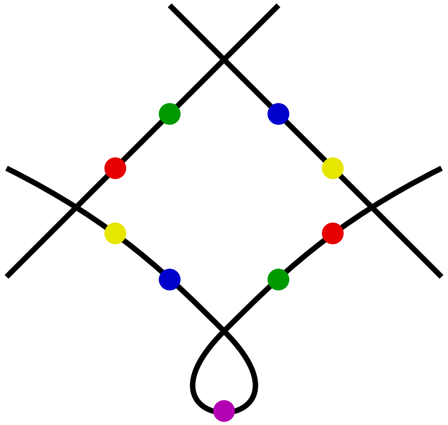


residue disks in $\mathrm{C}(\mathbb{Q}_p)$

special fiber of $\mathcal{C}$



$\mathcal{C}_s^{\mathrm{smooth}}(\mathbb{F}_p)$

Sketch of Proof (2)

Let $\rho\colon C(\mathbb{Q}_p) \to \mathcal{C}_s^{\mathsf{smooth}}(\mathbb{F}_p)$ be the reduction map.
Fix a residue disk $D = \rho^{-1}(\bar{P})$.
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Considering the Newton Polygons of w and ℓ, one shows that

$$\#\{P \in D : \langle \omega, i(P)\rangle = 0\} \le 1 + \nu(\omega, D) + \left\lfloor \frac{\nu(\omega, D)}{p-2} \right\rfloor,$$

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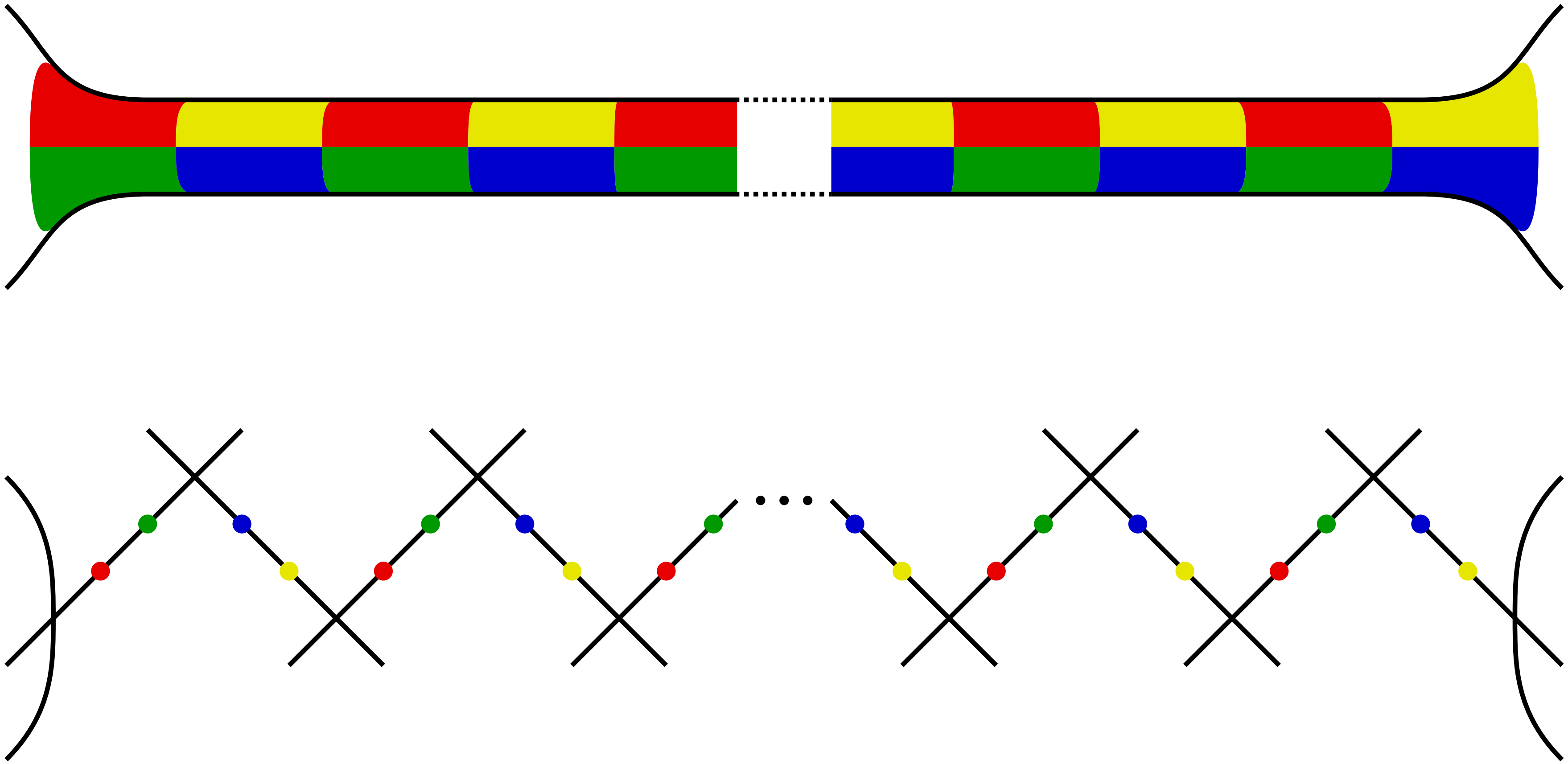
Picking an ‘optimal’ ω for each D and summing gives the result.

The Problem



How to Fix the Problem

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So we can cover $C(\mathbb{Q}_p)$ by $O(pg)$ disks D coming from points in $\mathcal{C}_s^{\text{smooth}}(\mathbb{F}_p)$ outside chains and by $O(g)$ annuli A coming from the chains.

The Key Result

Proposition.

For each annulus A there is a subspace $V_A \subset \Omega^1_C(\mathbb{Q}_p)$ with $\operatorname{codim} V_A \leq 2$ such that for $0 \neq \omega \in V_A$ satisfying a technical condition,

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Set $V_A = \{\omega \in \Omega^1_C(\mathbb{Q}_p) : a(\omega) = c(\omega) = 0\}$.

End of Proof of Main Result

Since $r \leq g - 3$, we have $V_\Gamma \cap V_A \neq \{0\}$ for all annuli A.
For C hyperelliptic (and p odd), an explicit computation shows that we can always pick a suitable $\omega \neq 0$ to get a bound

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Taking the 'optimal' ω for each annulus and for each disk and summing, we obtain the desired bound, which is of type

$$O(rg + pg) .$$

End of Proof of Main Result

Since $r \leq g-3$, we have $V_\Gamma \cap V_A \neq \{0\}$ for all annuli A.
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For a general p-adic field with ramification index $e < p-1$ and residue field of size q, the bound takes the shape

$$O\big(e(r+1)g + qg\big) .$$

Some Comments

- 'C hyperelliptic' and 'p odd' are used to describe $\omega|_A$ explicitly, allowing for bounding $\#(i^{-1}(\Gamma) \cap A)$ in terms of $\nu(\omega, A)$.

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- Taking $r = 0$, we obtain $\#i^{-1}(J(K)_{\text{tors}}) = O_K(g)$.

Improving the Poonen-Stoll 'One Point' Result

Theorem (Poonen-Stoll 2014).
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Theorem.
The 'probability' that an odd degree hyperelliptic curve of genus g over $\mathbb{Q}$ varying in any family defined by finitely many congruence conditions has the point at infinity as its only rational point is $\geq 1 - O(g^2 2^{-g})$.

Sketch of Proof

The key ingredient in the proof
(besides the work of Bhargava-Gross on 2-Selmer groups!)
was an estimate on the average size of the image of the '$\rho \log$ map'

$$\rho \log\colon C(\mathbb{Q}_2) \overset{i}{\hookrightarrow} J(\mathbb{Q}_2) \xrightarrow{\log_{\underline{\omega}}} \mathbb{Q}_2^g \dashrightarrow \mathbb{P}^{g-1}(\mathbb{Q}_2) \longrightarrow \mathbb{P}^{g-1}(\mathbb{F}_2)$$

where $\log_{\underline{\omega}}$ is $\log_J$ with respect to some basis $\underline{\omega}$ of $\Omega^1_{J/\mathbb{Q}_2} \cong T_0 J(\mathbb{Q}_2)^*$.

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Poonen-Stoll: The average size of the image is $O(g)$.

Our approach shows:
There is a uniform bound of type $O(g^2)$ on the size of the image.

Thank You!