Most odd degree hyperelliptic curves have only one rational point

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Odd Degree Hyperelliptic Curves

We consider hyperelliptic curves of genus g of the form

$$C\colon y^2 = f(x) = x^{2g+1} + c_1 x^{2g} + c_2 x^{2g-1} + \ldots + c_{2g} x + c_{2g+1}$$

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Let $\mathcal{F}_g = \{\underline{c} \in \mathbb{Z}^{2g+1} : \operatorname{disc}(f) \neq 0\}$ and $\mathcal{F}_{g,X} = \{\underline{c} \in \mathcal{F}_g : H(\underline{c}) < X\}$.

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For a subset $S \subset \mathcal{F}_g$, we define its lower and upper density by

$$\underline{\delta}(S) = \liminf_{X\to\infty} \frac{\#(S \cap \mathcal{F}_{g,X})}{\#\mathcal{F}_{g,X}}, \qquad \overline{\delta}(S) = \limsup_{X\to\infty} \frac{\#(S \cap \mathcal{F}_{g,X})}{\#\mathcal{F}_{g,X}}.$$

If $\underline{\delta}(S) = \overline{\delta}(S)$, then the common value is the density $\delta(S)$ of S.

The Meaning of the Title

$$C\colon y^2 = f(x) = x^{2g+1} + c_1 x^{2g} + c_2 x^{2g-1} + \ldots + c_{2g} x + c_{2g+1}$$

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Now the precise version of the statement in the title is as follows.

Theorem.
Let $\mathcal{C}_g$ be the subset of $\mathcal{F}_g$ consisting of curves C with $C(\mathbb{Q}) = \{\infty\}$.
Then $\underline{\delta}(\mathcal{C}_g) > 0$ for all $g \geq 3$ and

$$\lim_{g\to\infty} \underline{\delta}(\mathcal{C}_g) = 1\,.$$

More precisely, $\underline{\delta}(\mathcal{C}_g) = 1 - O\bigl(g2^{-g}\bigr)$.

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This is joint work with **Bjorn Poonen**.

The Selmer Group

Let J denote the Jacobian variety of C.
This is a g-dimensional abelian variety defined over $\mathbb{Q}$.
We take ∞ as base-point to embed C into J: $C \hookrightarrow J$ sends ∞ to 0.

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We take ∞ as base-point to embed C into J: $C \hookrightarrow J$ sends ∞ to 0.

Fix a prime p.
Then there is the p-Selmer group $\mathrm{Sel}_p J$ of J.
It is a computable finite abelian group of exponent p
that comes with homomorphisms

$$\frac{J(\mathbb{Q})}{pJ(\mathbb{Q})} \xhookrightarrow{\delta} \mathrm{Sel}_p J \xrightarrow{s} \frac{J(\mathbb{Q}_p)}{pJ(\mathbb{Q}_p)}$$

such that $s\delta$ is the homomorphsim induced by $J(\mathbb{Q}) \hookrightarrow J(\mathbb{Q}_p)$.

The Logarithm

$J(\mathbb{Q}_p)$ is a compact p-adic Lie group.

It has a logarithm

$$\log\colon J(\mathbb{Q}_p) \longrightarrow T_0 J(\mathbb{Q}_p) \cong \mathbb{Q}_p^g .$$

log is a homomorphism with finite kernel $J(\mathbb{Q}_p)_{\text{tors}}$.
Picking a suitable $\mathbb{Q}_p$-basis of the tangent space,
the image of log is $\mathbb{Z}_p^g$.

A Diagram

$\mathrm{C}(\mathbb{Q})$

A Diagram

$$\begin{array}{c} C(\mathbb{Q}) \\ \big\downarrow \\ J(\mathbb{Q}) \end{array}$$

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\end{array}$$

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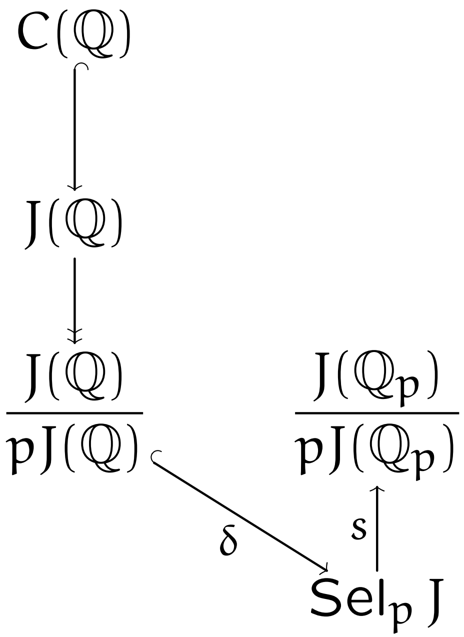

A Diagram

$$\begin{array}{ccc}
C(\mathbb{Q}) & \hookrightarrow & C(\mathbb{Q}_p) \\
\big\downarrow & & \big\downarrow \\
J(\mathbb{Q}) & \hookrightarrow & J(\mathbb{Q}_p) \\
\big\downarrow & & \big\downarrow \\
\dfrac{J(\mathbb{Q})}{pJ(\mathbb{Q})} & \longrightarrow & \dfrac{J(\mathbb{Q}_p)}{pJ(\mathbb{Q}_p)} \\
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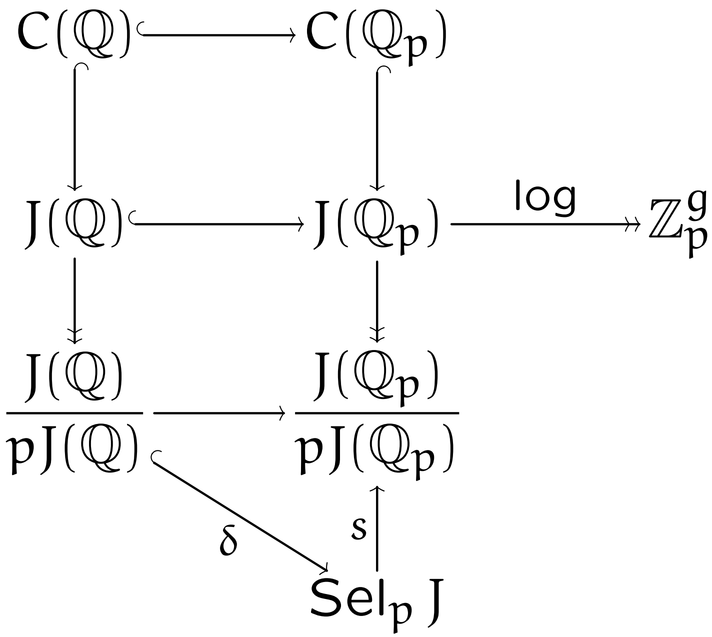

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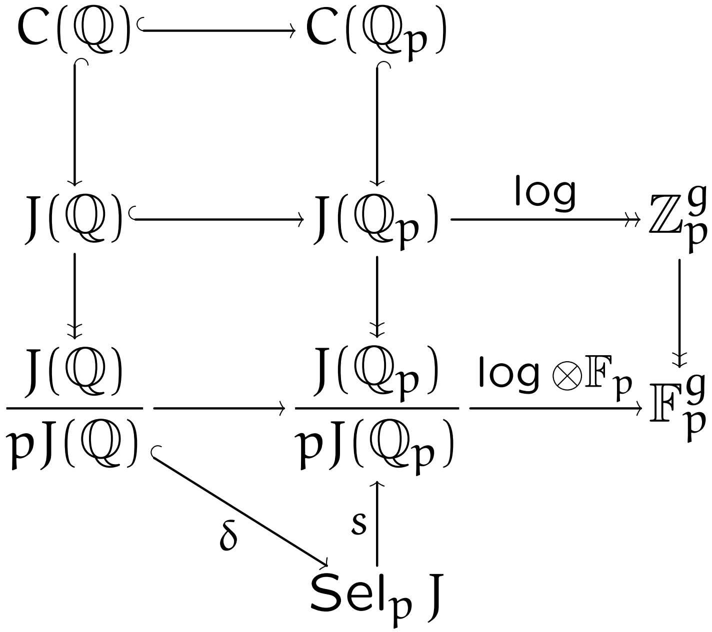
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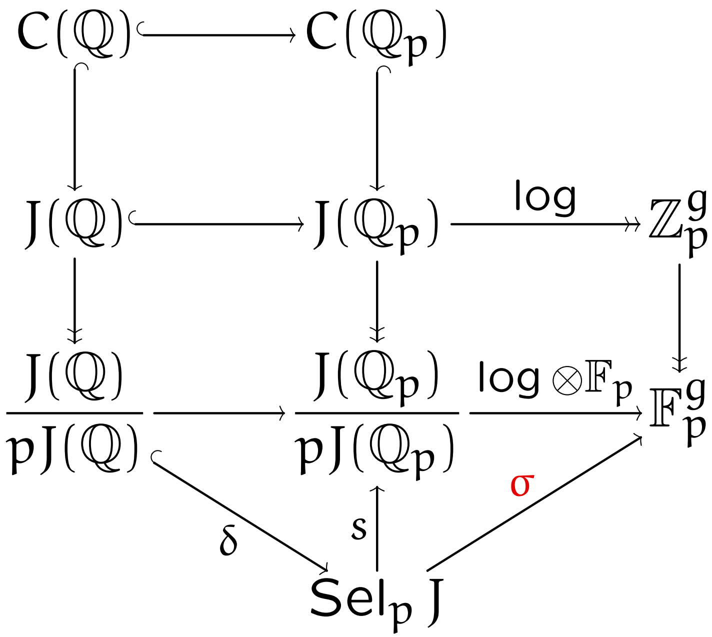
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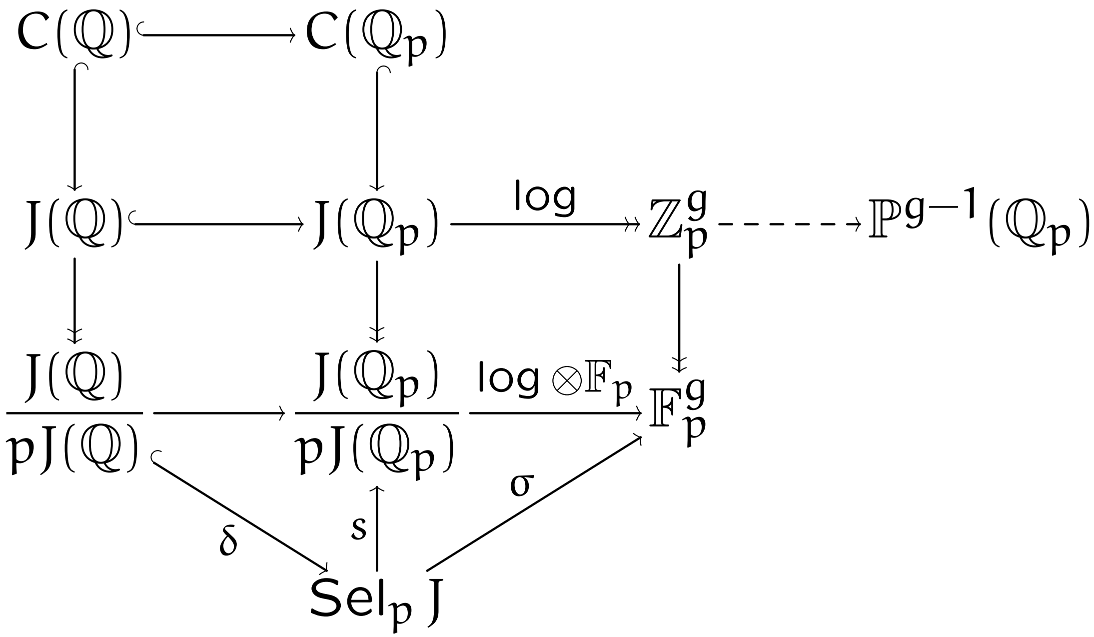

The dashed arrows are partially defined maps

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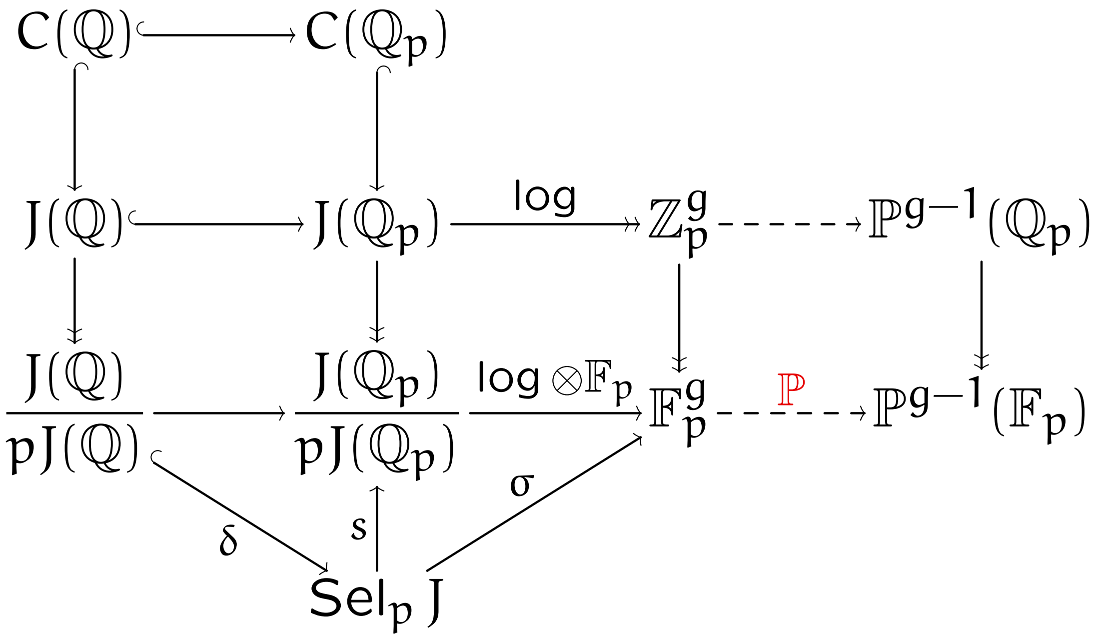

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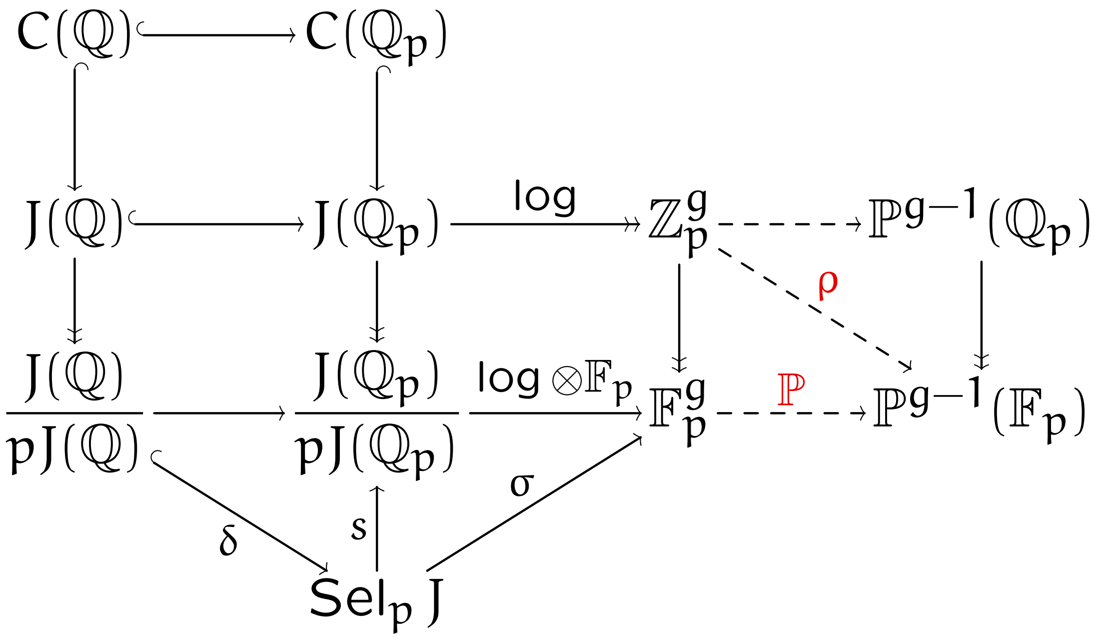

The dashed arrows are partially defined maps:
ρ is defined on $\mathbb{Z}_p^g \setminus \{0\}$, $\mathbb{P}$ is defined on $\mathbb{F}_p^g \setminus \{0\}$.

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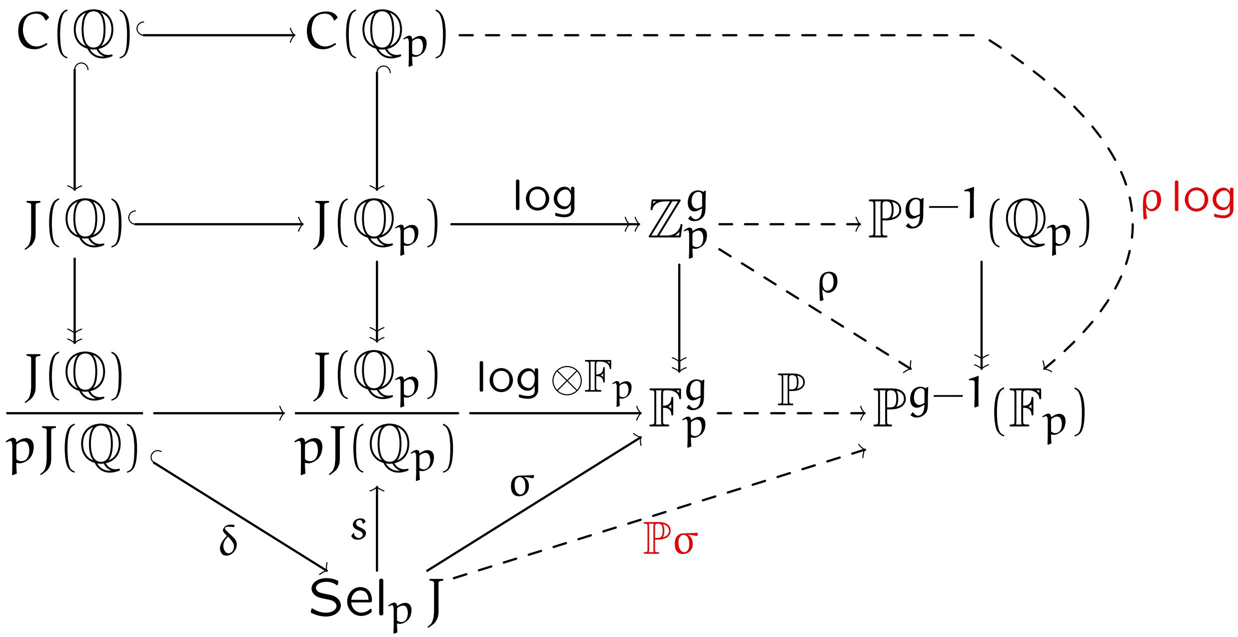
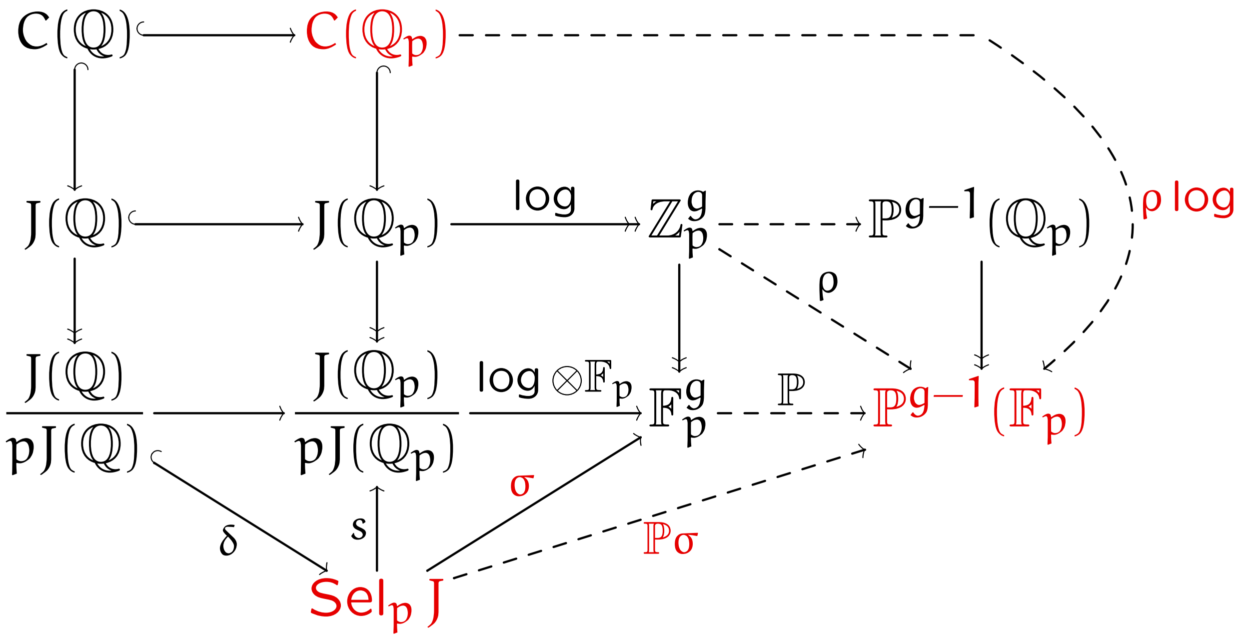


The dashed arrows are partially defined maps:

$\rho \log$ is defined on $C(\mathbb{Q}_p) \setminus J(\mathbb{Q}_p)_{\text{tors}}$, $\mathbb{P}\sigma$ is defined on $\text{Sel}_p J \setminus \ker \sigma$.

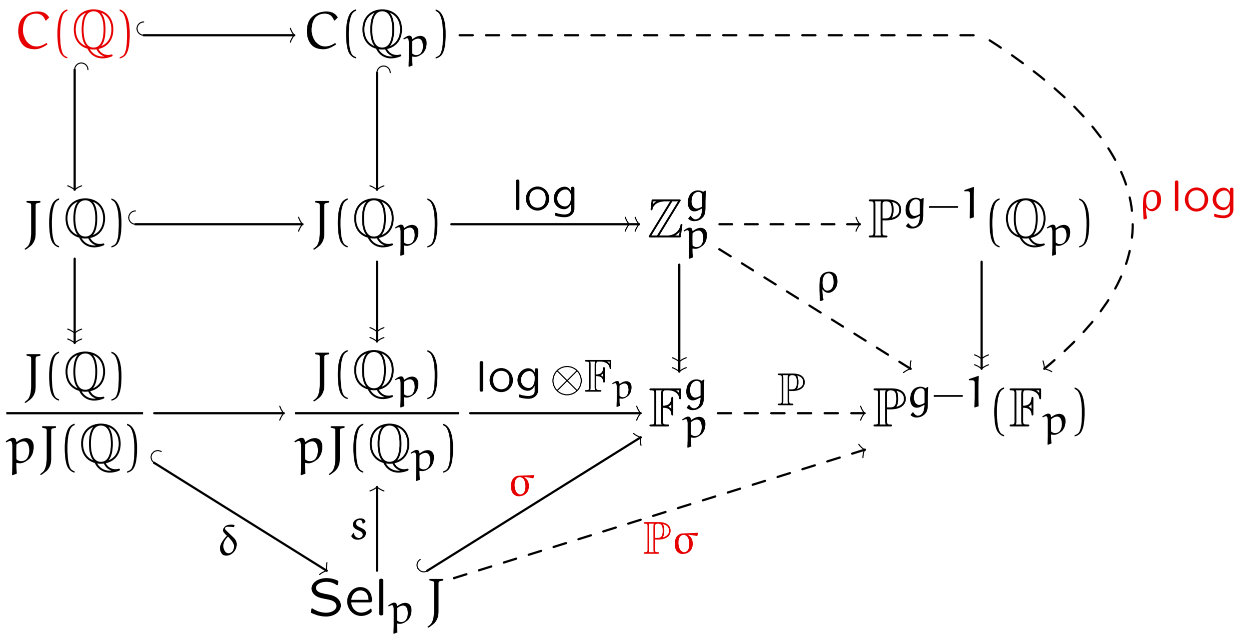
A Criterion



Proposition.

If σ is injective and $\rho \log(C(\mathbb{Q}_p))$ and $\mathbb{P}\sigma(\mathrm{Sel}_p J)$ are disjoint, then all points in $C(\mathbb{Q})$ are torsion points in J of order prime to p.

Proof



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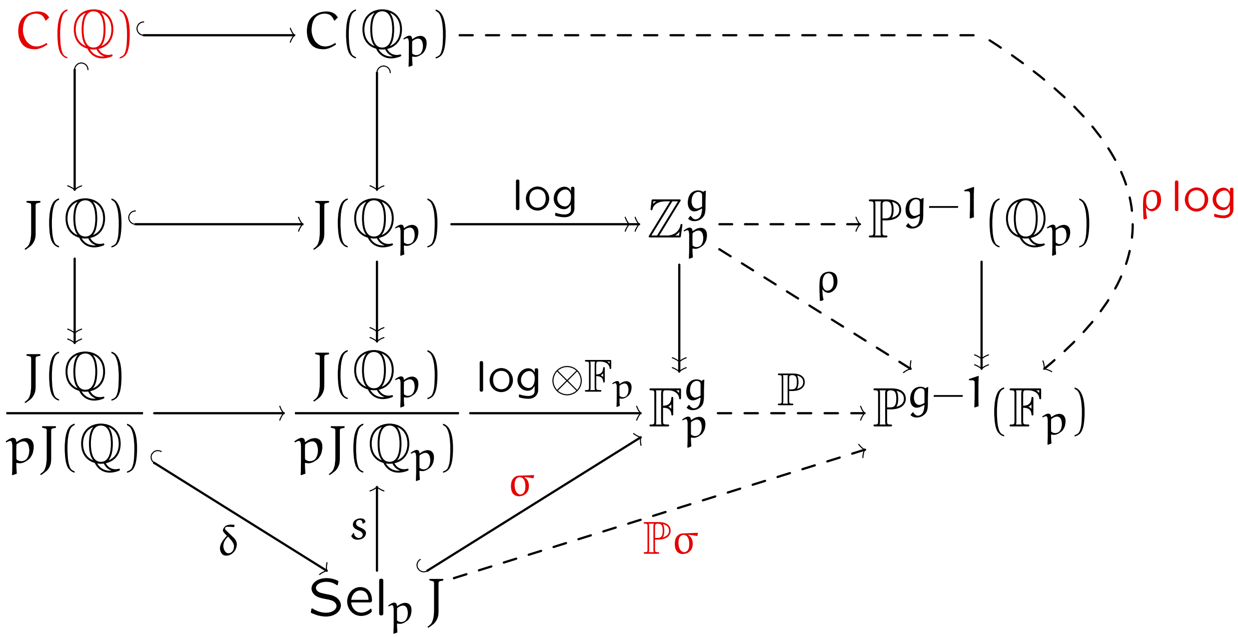

$P \in C(\mathbb{Q})$ not in $J(\mathbb{Q})[p']$ $\Longrightarrow$ $P = p^n Q$ with $Q \in J(\mathbb{Q}) \setminus pJ(\mathbb{Q})$, $n \geq 0$.

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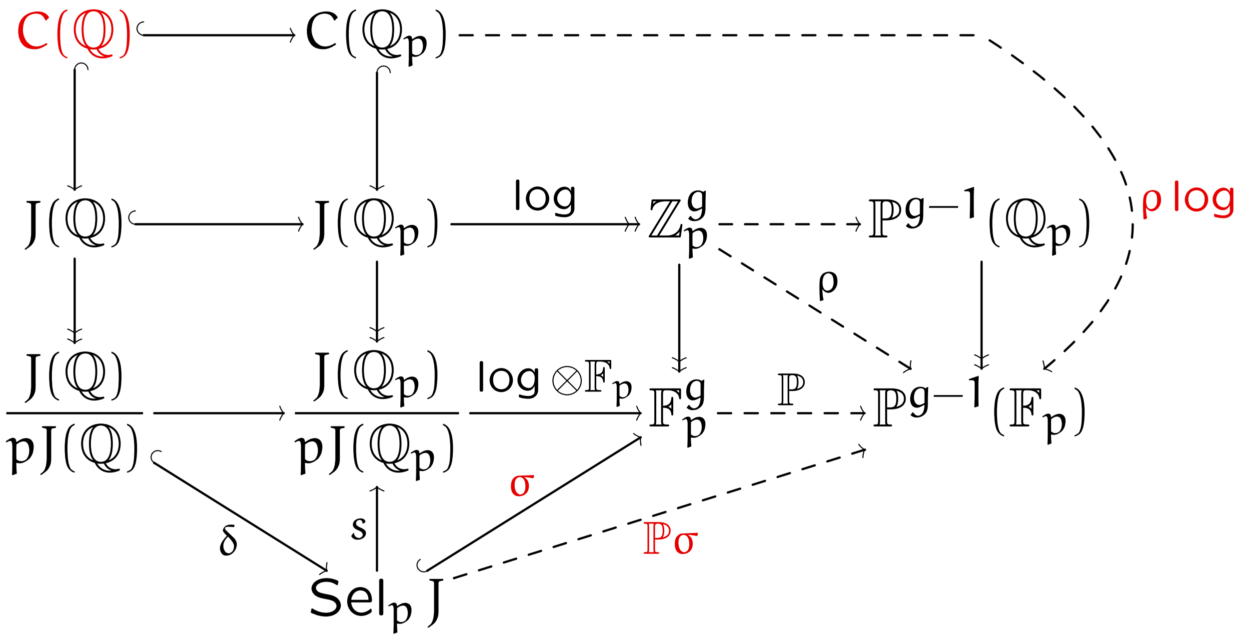

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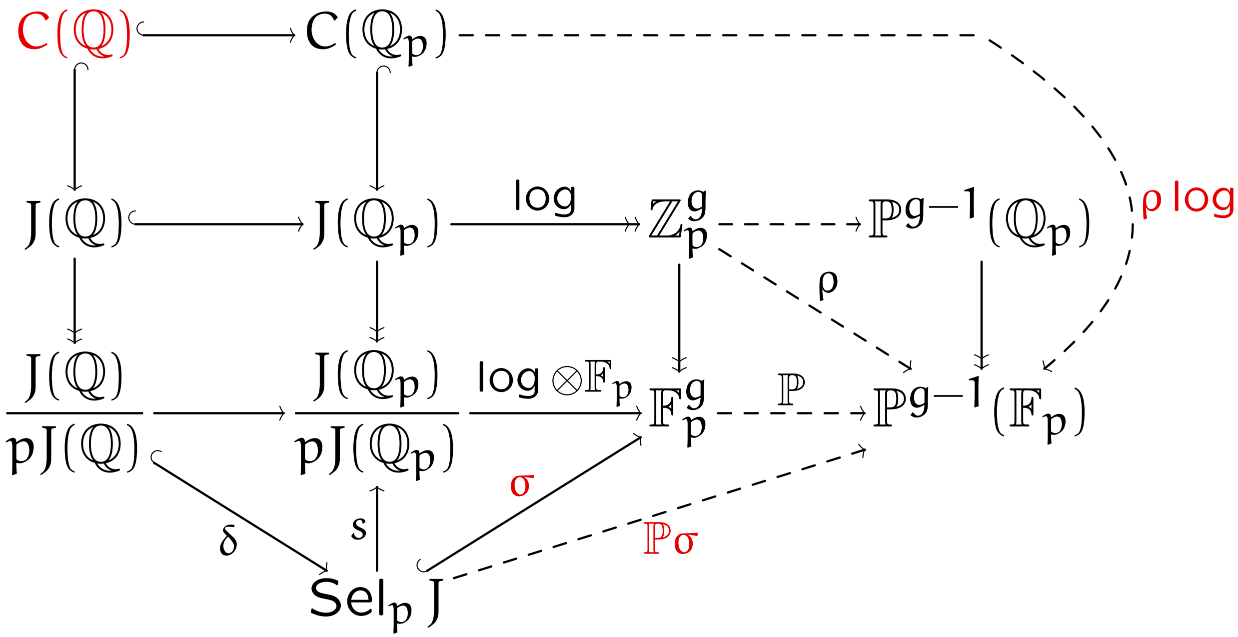

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σ injective $\implies \sigma\delta(Q + pJ(\mathbb{Q})) \neq 0$.

Now $\quad \mathbb{P}\sigma\big(\delta(Q + pJ(\mathbb{Q}))\big) = \rho \log(Q) = \rho \log(p^n Q) = \rho \log(P)$, $\quad$ contradiction!

Effectivity

Proposition.
If σ is injective and $\rho \log(C(\mathbb{Q}_p))$ and $\mathbb{P}\sigma(\mathrm{Sel}_p J)$ are disjoint,
then all points in $C(\mathbb{Q})$ are torsion points in J of order prime to p.

The conditions can be checked by a computation.

- The Selmer group is computable together with the map s.
- log is computable to any p-adic precision.
- $\implies \rho \log(C(\mathbb{Q}_p))$ can be computed.
- $\implies \sigma$ can be computed and checked for injectivity.
- $\implies \mathbb{P}\sigma(\mathrm{Sel}_p J)$ can be computed.

Bhargava-Gross

We now fix $p = 2$.

Manjul Bhargava and **Dick Gross** have recently proved the following.

Theorem.
The average of $\# \operatorname{Sel}_2 J$ exists in $\mathcal{F}_g$ and equals 3.
This is still true for subfamilies defined by congruence conditions.
If in such a subfamily $J(\mathbb{Q}_2)/2J(\mathbb{Q}_2) = G$ is constant,
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This implies that on 2-adically small subsets of $\mathcal{F}_g$,
an element of $\mathbb{F}_2^g \setminus \{0\}$ is in the image of σ with density $\le 2^{1-g}$
and that σ is not injective on a set of density $\le 2^{1-g}$.

Application of Bargava-Gross

We know:

If $J(\mathbb{Q})_{\mathrm{tors}} = 0$, σ is injective and $\rho \log(C(\mathbb{Q}_2)) \cap \mathbb{P}\sigma(\mathrm{Sel}_2 J) = \emptyset$,
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So we have to make sure that $\rho\log(C(\mathbb{Q}_2))$ is small on average;
then we will get $\rho\log(C(\mathbb{Q}_2)) \cap \mathbb{P}\sigma(\text{Sel}_2 J) = \emptyset$ for most curves.

Bounding $\rho \log(C(\mathbb{Q}_2))$

We can split $C(\mathbb{Q}_2)$ into a number of residue disks.
(A residue disk is a subset of the form $\{P \in C(\mathbb{Q}_2) : P \text{ reduces to } P_0\}$ for some point $P_0 \in \mathcal{C}^{\text{smooth}}(\mathbb{F}_p)$, where $\mathcal{C}$ is a regular model of C over $\mathbb{Z}_2$.)

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If D is a residue disk, then we can show that

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Let $d(C)$ denote the number of residue disks.
Since $\sum_D n_D \le 2g - 2$, we obtain

$$\#\rho\log\bigl(C(\mathbb{Q}_2)\bigr) \le 6(g-1) + 5d(C)\,.$$

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$\Longrightarrow$ In a 2-adic neighborhood of C in $\mathcal{F}_g$,
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To conclude the argument, we have to control $d(C)$.

Considering $d(C)$ as a random variable on $\mathcal{F}_g(\mathbb{Z}_2)$, we find:

Lemma.
The average of $d(C)$ is at most 3.

Result

$J(\mathbb{Q})_{\mathsf{tors}} = 0$ and σ injective and $\rho \log(C(\mathbb{Q}_2)) \cap \mathbb{P}\sigma(\mathsf{Sel}_2 J) = \emptyset \implies C(\mathbb{Q}) = \{\infty\}$.

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Conclusion.
The set of curves $C \in \mathcal{F}_g$ such that $C(\mathbb{Q}) \neq \{\infty\}$
has upper density $\leq (12g+20)2^{-g}$.

This is < 1 for $g \geq 7$ and tends to zero quickly as $g \to \infty$.

Small Genus

For each $g \geq 2$, we have $\#\rho\log(C(\mathbb{Q}_2)) = 1$
on a 2-adic neighborhood U of a curve isomorphic to

$$C_0\colon y^2 + y = x^{2g+1} + x + 1\,.$$

We get a relative lower density of curves $C \in \mathcal{F}_G \cap U$ with $C(\mathbb{Q}) = \{\infty\}$
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For $g \geq 3$, this is positive. Since $\delta(U) > 0$ as well, we obtain:

Conclusion.
For each $g \geq 3$,
the set of curves $C \in \mathcal{F}_g$ such that $C(\mathbb{Q}) = \{\infty\}$ has positive lower density.

Variants

For the family of hyperelliptic curves of genus g of the form

$$C \colon y^2 = f(x)$$

with f monic of degree $2g + 2$ (which have two rational points at infinity), **Shankar** and **Wang** have shown
(extending the methods of Bhargava-Gross and Poonen-St)
that the lower density of C with $\#C(\mathbb{Q}) = 2$ is $\geq 1 - (48g + 120)2^{-g}$.

For the family of general hyperelliptic curves of genus g
(f of degree $2g + 2$, not necessarily monic),
Bhargava, **Gross** and **Wang** have shown
that the lower density of C with $C(\mathbb{Q}) = \emptyset$ is $1 - o(2^{-g})$.

Heuristically, all these densities should be 1.