WHEN DO ALL ITERATES OF $x^2 + a$ HAVE MAXIMAL GALOIS GROUP?

MICHAEL STOLL

ABSTRACT. We provide an (in our opinion) simpler proof of the main results of [Sto92, Li20, Li21] on the question for which integers a the Galois groups of all iterates of x^2+a are maximal (equivalently, the associated arboreal Galois representation is surjective). At the same time, we extend the results to rational parameters a. The main result states that for a subset of rational numbers of relative density $1/2$ in the complement of a certain open interval, the maximality of the Galois group of the second iterate is sufficient for the arboreal Galois representation to be surjective. We also discuss experimental evidence suggesting that a similar statement involving the fourth iterate may hold in general.

The results were obtained in collaboration with Claude (Anthropic's AI model); they are also formalized in Lean (except for the experimental evidence).

1 Introduction

Let $a \in \mathbb{Q}$ be nonzero and consider the polynomial $f = f_a = x^2 + a \in \mathbb{Q}[x]$. Define its iterates by setting $f^{\circ 0} = x$ and, for $n \in \mathbb{Z}_{\ge 0}$,

$$f^{\circ(n+1)} = f(f^{\circ n}) = f^{\circ n}(x^2 + a) = (f^{\circ n})^2 + a\,.$$

Assuming that no iterate has multiple roots, we can arrange the pairs (n, ξ), where ξ is a root of $f^{\circ n}$ in some fixed algebraic closure of $\mathbb{Q}$, into an infinite complete binary tree T by decreeing that $(n+1, \eta)$ is a child of (n, ξ) when $\xi = f(\eta) = \eta^2 + a$. Then the absolute Galois group $\mathrm{Gal}_{\mathbb{Q}}$ of $\mathbb{Q}$ acts by tree automorphisms on T, giving a so-called *arboreal Galois representation*

$$\rho_a \colon \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}(T)\,.$$

We note that an iterate $f^{\circ n}$ of f has multiple roots if and only if zero is a root of $f^{\circ m}$ for some $1 \le m \le n$. Hence some iterate has multiple roots if and only if 0 is periodic under f. For $a \in \mathbb{Q}$, this occurs exactly when $a \in \{-1, 0\}$; see Remark 1.3 below.

We will discuss the following question.

Question 1.1. For which $a \in \mathbb{Q}$ is ρ_a surjective?

This is equivalent to asking that the action restricted to the levels up to index n, giving the finite tree T_n, be surjective for every n. The automorphism group of T_n is a Sylow 2-subgroup of the symmetric group S_{2^n} acting on the 2^n leaves of the tree. This group is isomorphic to the nth wreath power of the group S_2 with two elements. We will write $\rho_a^n \colon \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}(T_n)$ for the induced action.

This question was first asked by J. McKay in the special case $a = 1$; see [Odo88]. Odoni's earlier paper [Odo85] sets up some general theory. For an overview of what is known or expected on this and related questions, see the survey article [Jon13] by Jones.

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Definition 1.2. Define the sequence $(c_n)_{n\geq 0}$ (depending on a) of rational numbers as follows.

$$c_0 = 0\,, \qquad c_1 = -a\,, \qquad c_{n+1} = f(c_n) = c_n^2 + a \quad \text{for } n \geq 1.$$

This is the sequence obtained by iterating f starting with 0, *except* that we have changed the sign of c_1 (note that $f(-a) = f(a)$).

Remark 1.3. Note that $c_n = 0$ for some $n \geq 1$ holds if and only if 0 is periodic under f. For $a \in \mathbb{Q}$, this is the case if and only if $a \in \{-1, 0\}$. (This is well known; here is a quick sketch: if $a \notin \mathbb{Z}$, then the p-adic valuation doubles in each iteration step when p divides the denominator, so 0 cannot be periodic. So a must be an integer. For $a \geq 1$ or $a \leq -3$, the iteration sequence tends to infinity, leaving $a \in \{-2, -1, 0\}$. But for $a = -2$, 0 is preperiodic ($0 \mapsto -2 \mapsto 2 \mapsto 2$) and not periodic.)

We record how the p-adic valuation behaves along the sequence $(c_n)_{n\geq 1}$.

Lemma 1.4. *Let p be a prime and assume that a is a p-adic integer. Then c_n is a p-adic integer for all n, and in $\mathbb{Z}_p$, we have the congruence $c_{mk} \equiv c_m \bmod c_m^2$ for all $m \geq 2$ and $k \geq 1$, and similarly $c_k \equiv -c_1 = a \bmod c_1^2$ for all $k \geq 2$.*

Proof. That c_n is a p-adic integer for all n follows by an easy induction. Since $f = x^2 + a$ is even, so is $f^{\circ n}$ for all $n \geq 1$, so $f^{\circ n}(z) \equiv f^{\circ n}(0) \bmod z^2$ for all $z \in \mathbb{Z}_p$. Also, recall that $c_m = f^{\circ m}(0)$ when $m \geq 2$ and $c_1 = -f(0)$. We now fix $m \geq 2$ and do induction on k. The case $k = 1$ is trivial. For the induction step, note that

$$c_{m(k+1)} = f^{\circ(mk)}(c_m) \equiv f^{\circ(mk)}(0) = c_{mk} \bmod c_m^2\,.$$

The case $m = 1$ is similar, noting that $c_2 = f(c_1) \equiv f(0) = -c_1 \bmod c_1^2$. □

Lemma 1.5. *Let p be a prime and assume that $a \notin \{-1, 0\}$. Then exactly one of the following possibilities holds.*

(1) *For all $n \geq 1$, $v_p(c_n) = 0$.*

(2) *There are $m_p, e_p \in \mathbb{Z}_{\geq 1}$ such that*

$$v_p(c_n) = \begin{cases} 0 & \text{if } m_p \nmid n, \\ e_p & \text{if } n \geq 1 \text{ and } m_p \mid n. \end{cases}$$

(3) *There is $e_p \in \mathbb{Z}_{<0}$ such that $v_p(c_n) = 2^{n-1} e_p$ for all $n \geq 1$.*

Proof. If $v_p(c_1) = v_p(a) < 0$, then an easy induction shows that we are in the third case (with $e_p = v_p(a)$). If $v_p(a) \geq 0$, then we have from Lemma 1.4 that $v_p(c_n) \geq 0$ for all n. If $v_p(c_n) = 0$ for all $n \geq 1$, then we are in the first case. Otherwise, we are in the second case with $m_p = \min\{m \geq 1 : v_p(c_m) > 0\}$ and $e_p = v_p(c_{m_p})$; this follows from Lemma 1.4 with $m = m_p$ and the observation that the iteration sequence modulo p and starting from 0 is periodic with period m_p. See also the proof of [Sto92, Lemma 1.1(b)]. □

Definition 1.6. We say that a family $(r_i)_{i\in I}$ of rational numbers is *2-independent* if for every finite subset $S \subset I$, $\prod_{i\in S} r_i$ is a square only when $S = \emptyset$. Otherwise, the family is *2-dependent*.

(When all r_i are nonzero, this is equivalent to saying that the family of images of the r_i in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ is linearly independent over $\mathbb{F}_2$.)

Note that a family containing zero is always considered to be 2-dependent.

The following is the equivalence "a) $\Leftrightarrow$ b)" in part 1) of the Theorem in Section 1 of [Sto92].

Proposition 1.7. *Fix $a \in \mathbb{Q}$ and $n \in \mathbb{Z}_{\geq 1}$. Then ρ_a^n is defined and surjective if and only if $(c_1, \ldots, c_n)$ is 2-independent.*

Consequently, ρ_a is defined and surjective if and only if the family $(c_n)_{n\geq 1}$ is 2-independent.

Note that [Sto92] defines "2-independent" only for families of nonzero numbers and has the general assumption that $-a$ is not a square (which implies that all c_n are nonzero). However, the statement remains true in the more general version stated above: when $-a$ is a square, ρ_a^1 is not surjective (hence ρ_a^n is not surjective for $n \geq 1$) and $c_1 = -a$ is a square, so $(c_1, \ldots, c_n)$ is 2-dependent. This includes the cases $a = 0$ and $a = -1$, where ρ_a is not defined.

So our approach will be to show that $(c_n)_{n\geq 1}$ is 2-independent for $a \in \mathbb{Q}$ satisfying certain conditions.

The proofs in [Sto92, Li20, Li21] follow [Odo88] in that they consider instead the sequence (b_n) (notation from [Sto92]; Odoni writes $c(n)$ and $a(n)$ for our c_n and b_n) obtained from (c_n) by multiplicative Möbius inversion. When $a \in \mathbb{Z} \setminus \{-1, 0\}$, then (c_n) is a rigid divisibility sequence in $\mathbb{Z}$ (see Lemma 1.5), which implies that the b_n are pairwise coprime integers (see also [Odo88, Lemma 4.4] or [Sto92, Lemma 1.1]). The sequence $(c_n)_{n\geq 1}$ is 2-independent if and only if $(b_n)_{n\geq 1}$ is, and the latter follows if $|b_n|$ is never a square for $n \geq 2$ and b_1 is not a square. Trading (c_n) for (b_n) gives a simpler criterion to check — we only need to look at individual terms instead of arbitrary finite products — but at the price of having to work with more complicated objects.

One purpose of this note is to show that we can as easily (or even more easily) obtain the results of [Sto92, Li20, Li21] on when ρ_a is surjective by working directly with (c_n).

The second purpose is to generalize the results that so far cover certain subsets of integers a to include honest rational numbers. The main result is that for essentially half of all rational numbers outside the open interval $(-2, \alpha)$, where α is the unique real root of the polynomial $x^3 + 2x^2 + 2x + 2$, ρ_a is surjective as soon as (and therefore if and only if) ρ_a^2 is surjective.

By Proposition 1.7, the latter means that $(c_1, c_2) = (-a, a^2 + a)$ is 2-independent; concretely, that none of $-a$, $a^2 + a$, and $-a(a^2 + a)$ is a square. The word "half" above refers to six of the twelve possible pairs of residue classes modulo 4 of the numerator and denominator of a (assumed to be in lowest terms with positive denominator), and "essentially" means that the statement holds for all such a outside $(-2, \alpha')$, where $\alpha' > \alpha$ is another algebraic number; for $a \in (\alpha, \alpha')$, we can still prove it outside a density-zero subset of the half we are considering (and for all values of a in this density-zero subset that we checked, the claim could be verified by a finite computation). This "half" can also be interpreted as a relative density: we order the rational numbers $a = r/s$ outside the interval $(-2, \alpha)$ by their height $\max\{|r|, s\}$; then the proportion of the numbers of height bounded by X that are covered by our results tends to $1/2$ as X tends to infinity.

See Section 6 for precise statements. We extend these results to two quadratic families in Section 7; this generalizes results from [Li21]. In Section 8 we add a discussion of the conjecture that surjectivity of ρ_a^N for some fixed (sufficiently large) N should

be sufficient for surjectivity of ρ_a; our new contribution is to extend the experimental evidence to all $a \in \mathbb{Q}$ of height up to 10^6 and quadratic relations among $(c_1, \dots, c_{20})$. The results support such a conjecture with $N = 4$.

A formalization in Lean (with Mathlib) of the results in this paper is available on GitHub [Sto26, Part I of the blueprint].

AI declaration

Work on this paper and its formalization was done in collaboration with Claude, Anthropic's AI model (in various incarnations: Opus 4.8, Opus 5, Opus 5.5; Fable 5, Fable 5.1). This proceeded in the following steps.

1. When given test access to an autoformalization system by Logos Research in early July 2026, I asked it to formalize [Sto92] (my very first published paper, which I had been planning to get formalized for some time).
2. I worked quite intensely with Claude over an extended period of time on improving the resulting initial formalization and on including the results of [Li20, Li21]. The result is available at [Sto26, Part II of the blueprint].
3. This renewed my interest in the question, so I asked Claude to do a literature search to find out whether there are any further results available on the question considered here, in particular for rational parameters. This search came up essentially empty.
4. I then asked Claude to see if it could use the method from [Sto92] (and its extensions by Li) to deal with more cases of integers and also to deal with rational parameters. Claude could extend one of Li's results to, in a sense, twice as many parameters. It also produced a theorem on rational parameters, which had a rather technical statement. At the same time, Claude extended the formalization to cover the new results.
5. I then attempted to digest this by writing it up in my own words. While doing this, I realized that one can simplify the proofs by working with the sequence (c_n) directly rather than taking a detour via the multiplicatively Möbius-inverted sequence (b_n). I also noticed that one can control the sign of $c_n + c_{n+1}$ in an interval quite a bit to the left of -1 (Claude's result required $a \le -2$ or $a > -1$).
6. I enlisted Claude's help again. It reviewed my notes, corrected several errors, and contributed further ingredients that are now part of this paper: the sign pattern for $\alpha' < a < -1$ and criterion (T4), the criteria (Z3), (T3) and those at the primes dividing r or s, the character τ, Lemmas 7.1 and 7.2, and the extension of the two quadratic families to rational ℓ. Prompted by me, it checked that the new approach recovers its earlier results. Prompted by my further questions, it found that outside $(\alpha, \alpha']$ the conditions reduce to the 2-independence of c_1 and c_2 (Theorem 6.1(1)), and worked out the classes in Theorem 6.1(2)(a)–(d). It also formalized all results of this paper [Sto26] and made Figure 1.
7. I then modified and extended the notes I had written, resulting in a draft of the present paper (using Claude for proof-reading).
8. Finally, an audit of the formalization by Claude led to some further improvements, which I incorporated.
9. Claude also helped with setting up a patched version of `ratpoints` and running it on a number of hyperelliptic curves to obtain the experimental evidence in Section 8.

The text of this paper was written entirely by me (in some cases using text produced by Claude as a starting point). I take responsibility for all the claims made.

2 Quadratic characters

We will now show how to obtain surjectivity of ρ_a without considering the Möbius-inverted sequence (b_n). This approach will work for rational a with certain properties; the results of [Sto92, Li20, Li21] will follow as special cases when restricting a to be an integer.

Recall that we need to show that the family $(c_n)_{n\geq 1}$ is 2-independent. To detect 2-independence, we will use group homomorphisms $\mathbb{Q}^\times \to \{\pm 1\}$.

Definition 2.1. We define the following homomorphisms $\mathbb{Q}^\times \to \{\pm 1\}$. When p is a prime and $z \in \mathbb{Q}^\times$, we let z_p denote an integer whose image in $\mathbb{Z}/p\mathbb{Z}$ (or $\mathbb{Z}/8\mathbb{Z}$ when $p = 2$) is the same as that of $p^{-\nu_p(z)} \cdot z$.

(1) sign is the sign map;
(2) $\mathrm{val}_p \colon z \mapsto (-1)^{\nu_p(z)}$, where p is a prime and ν_p is the (additive) p-adic valuation;
(3) $\chi_p \colon z \mapsto \left(\frac{z_p}{p}\right)$, where p is an odd prime, i.e., χ_p is induced by the Legendre symbol mod p on the prime-to-p part;
(4) $\chi_4 \colon z \mapsto (-1)^{(z_2-1)/2}$ and $\chi_8 \colon z \mapsto (-1)^{(z_2^2-1)/8}$; these are induced by the usual Dirichlet characters modulo 4 and modulo 8 applied to the odd part.

Note that the class of z modulo squares is already determined by the values of sign and all the val_p on z. It is, however, helpful to use the characters that are induced by Dirichlet characters as well.

Recall from Remark 1.3 that c_n is nonzero for all $n \geq 1$ if and only if $a \in \mathbb{Q} \setminus \{-1, 0\}$.

Definition 2.2. Let $a \in \mathbb{Q} \setminus \{-1, 0\}$ and $n \geq 1$. We say that a *has* ϕ_n if there is a character $\phi_n \colon \mathbb{Q}^\times \to \{\pm 1\}$ such that $\phi_n(c_n) = -1$ and $\forall m > n \colon \phi_n(c_m) = 1$.

We have the following easy consequence.

Proposition 2.3. *Let $a \in \mathbb{Q} \setminus \{-1, 0\}$. Then ρ_a is surjective if and only if a has ϕ_n for all $n \geq 1$. If ρ_a is surjective, then for every sequence $(\delta_n)_{n\geq 1}$ of signs, there is a character $\chi \colon \mathbb{Q}^\times \to \{\pm 1\}$ such that $\chi(c_n) = \delta_n$ for all $n \geq 1$.*

Proof. We recall that ρ_a is surjective if and only if $(c_n)_{n\geq 1}$ is 2-independent.

First assume that a has ϕ_n for all $n \geq 1$; we show that $(c_n)_{n\geq 1}$ is 2-independent. Let $S \subset \mathbb{Z}_{\geq 1}$ be finite and nonempty and set $n := \min S$. By assumption, there is $\phi_n \colon \mathbb{Q}^\times \to \{\pm 1\}$ such that $\phi_n(c_n) = -1$ and $\phi_n(c_m) = 1$ for all $m > n$. This implies that

$$\phi_n\Big(\prod_{k\in S} c_k\Big) = \phi_n(c_n) \cdot \prod_{k\in S, k>n} \phi_n(c_k) = -1\,;$$

in particular, $\prod_{k\in S} c_k$ cannot be a square.

Conversely, assume that ρ_a is surjective, so that $(c_n)_{n\geq 1}$ is 2-independent. Then the images of the c_n in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ are linearly independent over $\mathbb{F}_2$, so they can be extended to a basis. Given signs δ_n, there is therefore a character $\mathbb{Q}^\times/\mathbb{Q}^{\times 2} \to \{\pm 1\}$ sending the image of c_n to δ_n for each $n \geq 1$. This gives in particular characters that show that a has ϕ_n for every $n \geq 1$. □

Remark 2.4. Alternatively, we could say that a *has* ϕ'_n when there is a character $\phi'_n\colon \mathbb{Q}^\times \to \{\pm 1\}$ such that $\phi'_n(c_m) = 1$ for all $1 \le m < n$ and $\phi'_n(c_n) = -1$. Then it follows that $(c_n)_{n\ge 1}$ is 2-independent if a has ϕ'_n for all $n \ge 1$ (by considering $\max S$ instead of $\min S$). For example, $a \in \mathbb{Z} \setminus \{-1, 0\}$ has ϕ'_n if c_n has a primitive prime divisor p (i.e., a prime divisor not dividing any c_m for $1 \le m < n$) such that $\nu_p(c_n)$ is odd (then $\phi'_n = \mathrm{val}_p$ will do). It is known that for integral $a \notin \{-2, -1, 0\}$, c_n has a primitive prime divisor p whenever $n \ge 2$ [Ric07, DH12] (and, more generally, for rational $a = r/s \notin [-2, -1]$ except for $n = 2$ when $r + s = 1$ [Kri13]), but it is rather hard to directly control the parity of $\nu_p(c_n)$. For example, when $a = 3$, we have $c_1 = -3$, $c_2 = 3 \cdot 2^2$, $c_3 = 3 \cdot 7^2$, so the primitive prime divisors 2 for $n = 2$ and 7 for $n = 3$ occur with even valuation (and $c_2c_3 = 42^2$ is a square, so (c_n) is 2-dependent in this case). In contrast to this, it will turn out that we are actually able to show that a has ϕ_n for all $n \ge 1$, at least for certain a.

To help with constructing characters ϕ_n later, it is useful to introduce two further kinds of characters.

Definition 2.5. Let $a \in \mathbb{Q} \setminus \{-1, 0\}$.

(1) We say that a *has* ϕ_0 if there is a character $\phi_0\colon \mathbb{Q}^\times \to \{\pm 1\}$ such that $\phi_0(c_n) = -1$ for all $n \ge 1$.

(2) We say that a *has* τ ("tail") if there is a character $\tau\colon \mathbb{Q}^\times \to \{\pm 1\}$ such that $\tau(c_1) = 1$ and $\forall m \ge 2\colon \tau(c_m) = -1$.

By Proposition 2.3, a has ϕ_0, ϕ_1, and τ when ρ_a is surjective. Note that when a has any two of ϕ_0, ϕ_1, and τ, then a also has the third (a suitable character can be obtained as the product of the two other characters). Similarly, we have the following.

Lemma 2.6. *Let* $a \in \mathbb{Q} \setminus \{-1, 0\}$ *and let* $n \ge 1$. *If* a *has* ϕ_0 *and there exists a character* $\psi\colon \mathbb{Q}^\times \to \{\pm 1\}$ *such that* $\psi(c_{n+m}) = -\psi(c_n)$ *for all* $m \ge 1$, *then* a *has* ϕ_n.

Proof. We can take $\phi_n = \psi$ or $\phi_n = \phi_0 \cdot \psi$, depending on whether $\psi(c_n) = -1$ or 1. □

Here is a slightly more involved way of obtaining ϕ_n.

Lemma 2.7. *Let* $n \ge 1$. *Assume that* $a \in \mathbb{Q} \setminus \{-1, 0\}$ *has* ϕ_0, *that some character* ψ *satisfies* $\psi(c_{m+1}) = -\psi(c_m)$ *for all* $m \ge n$, *and that some character* χ *satisfies* $\chi(c_n) = x$, $\chi(c_{n+2k}) = -x$ *for* $k \ge 1$ *and* $\chi(c_{n+2k+1}) = y$ *for* $k \ge 0$, *with* $x, y \in \{\pm 1\}$. *Then* a *has* ϕ_n.

Proof. If $y = -x$, then χ is x at c_n and $-x$ at all later terms. If $y = x$, then the character $\chi \cdot \psi$ has the same property with $x\psi(c_n)$ in place of x. In both cases Lemma 2.6 applies. □

In Section 4, we will use Lemma 2.6 to show that, for the "half" of rational numbers mentioned in the introduction, a has ϕ_n for all $n \ge 2$, provided that a has ϕ_0. We will then consider sufficient conditions for the existence of ϕ_0, ϕ_1, and τ. Combining these with Proposition 2.3 will give our main result.

3 Sign information

We record information on the sign of c_n and of $c_{n+1} + c_n$.

Lemma 3.1. (1) *If* $a > 0$, *then* $c_1 = -a < 0$ *and* $c_n > 0$ *for all* $n \geq 2$. *In particular,* a *has* $\phi_1 = \mathrm{sign}$.

(2) *If* $-1 < a < 0$, *then* $c_1 = -a > 0$ *and* $c_n < 0$ *for all* $n \geq 2$. *In particular,* a *has* $\tau = \mathrm{sign}$.

(3) *If* $a \leq -2$, *then* $c_n \geq -a > 0$ *for all* $n \geq 1$.

Proof.

(1) We have $c_{n+1} = c_n^2 + a > 0$ for all $n \geq 1$.

(2) We have $c_2 = a^2 + a = a(a+1) < 0$. Assuming that $c_n < 0$, we see that $c_{n+1} = c_n^2 + a \leq a^2 + a < 0$ for all $n \geq 2$; here we use that $c_n = c_{n-1}^2 + a \geq a$, so (using $c_n < 0$) $c_n^2 \leq a^2$.

(3) We have $c_1 = -a > 0$. Assuming $c_n \geq -a$, we obtain $c_{n+1} = c_n^2 + a \geq a^2 + a \geq -a > 0$ for all $n \geq 1$. □

Definition 3.2. We write α for the unique real root of $x^3 + 2x^2 + 2x + 2$; note that $\alpha \approx -1.543689$ and that $a > \alpha$ is equivalent to $a^3 + 2a^2 + 2a + 2 > 0$.

Lemma 3.3. *Assume that* $a \in (\alpha, 0)$. *Then* $c_{n+1} + c_n < 0$ *for all* $n \geq 2$.

Proof. First, for $-2 \leq a \leq 0$ we have that $c_3 \leq c_2$ (since $c_3 - c_2 = a^3(a+2)$). This implies that the interval $[a, a^2 + a]$ is stable under f_a. (Note that f is convex from below, so it takes its maximum on any interval at one of the endpoints. Since $c_2 \geq c_3$, this maximum is c_2.) So in particular, we have $a \leq c_n \leq c_2$ for all $n \geq 2$. The conditions $a^3 + 2a^2 + 2a + 2 > 0$ and $a < 0$ imply $c_2^2 + c_2 + a < 0$. Fix $n \geq 2$. If $c_n \geq -1/2$, then

$$c_{n+1} + c_n = c_n^2 + c_n + a \leq c_2^2 + c_2 + a < 0\,.$$

Otherwise,

$$c_{n+1} + c_n = c_n^2 + c_n + a \leq a^2 + a + a = a(a+2) < 0\,. \qquad \square$$

Note that for $a = \alpha$, we obtain $c_3 + c_2 = 0$, so the condition in the lemma is sharp.

There is another interval where we can control the sign of c_n in a useful way.

Definition 3.4. We write α' for the larger real root of $x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1$. We have $\alpha' \approx -1.310703$ and $\alpha' < a < -1$ is equivalent to $c_3 < 0 < c_4$. (c_4 is $a(a+1)$ times the polynomial evaluated at a.)

Lemma 3.5. *Assume that* $\alpha' < a < -1$. *Then for all* $k \geq 1$: $0 < c_4 \leq c_{2k} \leq c_2$ *and* $a \leq c_{2k+1} \leq c_3 < -1$. *So* $\big(\mathrm{sign}(c_n)\big)_{n\geq 1} = (1, 1, -1, 1, -1, 1, \ldots)$.

Proof. $c_4 = c_3^2 + a > 0$ gives $c_3^2 > -a > 1$, so $c_3 < -1$; and $c_2 = a(a+1) > 0$. On $B_1 = [a, c_3] \subset (-\infty, 0)$ the map f is decreasing, so $f(B_1) = [c_4, c_2]$, using $f(a) = c_2$; on $B_2 = [c_4, c_2] \subset (0, \infty)$ it is increasing, so $f(B_2) = [c_5, c_3] \subseteq B_1$, as $c_5 \geq a$. Also $c_2 \in B_2$ (as $c_3 \in [a, 0)$ gives $c_4 \leq a^2 + a$) and $c_3 \in B_1$; induction on k. □

So under the assumption of the lemma, sign can serve as the character ψ in Lemma 2.7 when $n \geq 2$.

4 ϕ_n for $n \geq 2$

From now on, we write $a = r/s$ with $r, s \in \mathbb{Z}$, $s > 0$ and $\gcd(r, s) = 1$.

In this section, we will show that under suitable conditions (r and s satisfy some congruence conditions modulo 4 and a is not in a "bad" interval), a will have ϕ_n for all $n \geq 2$, assuming it also has ϕ_0. The latter is needed to be able to apply Lemma 2.6.

The first step is to write c_n as a fraction in lowest terms. Lemma 1.5 shows that the denominator of c_n is $s^{2^{n-1}}$ and its numerator is divisible by r. So we can write $c_n = rw_n/s^{2^{n-1}}$, where

$$w_1 = -1 \qquad \text{and} \qquad w_{n+1} = rw_n^2 + s^{2^n-1}$$

are integers. The first few terms are

$$w_1 = -1\,, \quad w_2 = r + s\,, \quad w_3 = r^3 + 2r^2s + rs^2 + s^3\,, \quad \dots\,.$$

Since r and s are coprime, w_n is coprime to r and s, which also implies that w_{n+1} and w_n are coprime. Considered as a polynomial in r and s, w_n is a binary form of degree $2^{n-1} - 1$ with integral coefficients; it is the homogenization of c_n/a, considered as a polynomial in a.

We note that the numerator of $c_{n+1} + c_n$ is r times

$$v_n := rw_n^2 + w_n s^{2^{n-1}} + s^{2^n-1} = w_{n+1} + w_n s^{2^{n-1}}\,.$$

The first two terms are

$$v_1 = r \qquad \text{and} \qquad v_2 = r^3 + 2r^2s + 2rs^2 + 2s^3\,.$$

For $a > \alpha$ the number v_n is positive when $n \geq 2$ (by Lemma 3.3 when $a < 0$ and obviously so when $a \geq 0$). On the other hand, v_n is negative when $a \leq -2$. Since w_{n+1} and w_n are coprime, v_n is coprime to w_n, and since w_{n+1} is coprime to s, v_n is also coprime to s; it follows that v_n is coprime to w_{n+1} as well.

Definition 4.1. We set

$$\mathcal{A} = \left\{a \in \mathbb{Q} \setminus \{-1, 0\} : a \leq -2 \text{ or } a > \alpha\right\}.$$

Definition 4.2. Let $a \in \mathcal{A}$. We set

$$\varepsilon = \varepsilon_a = \begin{cases} 1 & \text{if } a > \alpha, \\ -1 & \text{if } a \leq -2. \end{cases}$$

Then the sign of v_n for $n \geq 2$ is exactly ε_a. Note that $v_2 = s^3(a^3 + 2a^2 + 2a + 2)$, which changes sign at α, so we lose control over this sign when a passes α from above. This is the reason why the interval $(-2, \alpha)$ is excluded from our results.

Lemma 4.3. *Let $a = r/s \in \mathcal{A}$ and assume that a has ϕ_0.*

(1) *Fix $n \geq 2$. Then a has ϕ_n if v_n has a prime divisor $p \equiv 3 \bmod 4$ (so in particular when the odd part of $|v_n|$ is $\equiv 3 \bmod 4$).*

(2) *In each of the following cases, a has ϕ_n for all $n \geq 2$.*

(C1) *s is odd and $r + 2s \equiv -\varepsilon_a \bmod 4$ (equivalently, $r \equiv \varepsilon_a \bmod 4$).*

(C2) *s is odd and $r + s \equiv -\varepsilon_a \bmod 4$.*

(C3) *s is even and $r \equiv -\varepsilon_a \bmod 4$.*

Proof.

(1) Let $p \equiv 3 \bmod 4$ be a prime divisor of $v_n = w_{n+1} + w_n s^{2^{n-1}}$. The considerations above then show that $p \nmid w_n, w_{n+1}, s$. It also follows that $p \nmid r$: $w_m \equiv s^{2^{m-1}-1} \bmod r$ for $m \geq 2$, so $v_n \equiv 2s^{2^n-1} \bmod r$. This implies that c_n and c_{n+1} are p-adic units and $c_{n+1} \equiv -c_n \bmod p$. An easy induction then shows that $c_m \equiv c_{n+1} \bmod p$ for all $m > n$. Since $p \equiv 3 \bmod 4$, we have $\chi_p(-1) = -1$. Together, these observations show that $\chi_p(c_m) = -\chi_p(c_n)$ for all $m > n$. Now apply Lemma 2.6.

(2) We show that in each case, the odd part of $|v_n|$ is $\equiv 3 \bmod 4$ for all $n \geq 2$; then part (1) applies.

First assume that s is odd. Then $s^{2^{m-1}} \equiv 1 \bmod 8$ for all $m \geq 2$, which implies that $v_n \equiv w_{n+1} + w_n \bmod 8$. If r is odd, we see easily that $w_{2m} \equiv r + s \bmod 4$ and $w_{2m+1} \equiv s \bmod 4$ for all $m \geq 1$. This implies that $v_n \equiv r + 2s \bmod 4$ and hence that $|v_n| \equiv \varepsilon_a(r+2s) \bmod 4$. So when $r+2s \equiv -\varepsilon_a \bmod 4$, it follows that $|v_n| \equiv 3 \bmod 4$.

If r is even, we find that $w_m \equiv r+s \bmod 8$ for all $m \geq 2$. Then $v_n \equiv 2(r+s) \bmod 8$, hence $|v_n| \equiv \varepsilon_a \cdot 2(r+s) \bmod 8$. When $r + s \equiv -\varepsilon_a \bmod 4$, this is $\equiv 6 \bmod 8$, so its odd part is $\equiv 3 \bmod 4$.

Finally, assume that s is even. Then $v_m \equiv w_{m+1} \bmod 4$ for all $m \geq 2$. We also have that $w_m \equiv r \bmod 4$ for all $m \geq 3$, and we conclude in the same way as before. □

We note that for each possible sign ε_a, the three cases listed cover six of the possible twelve primitive residue classes modulo 4 for pairs (r, s). In this sense, Lemma 4.3 applies to half of the rational numbers in $\mathcal{A}$.

When (r, s) does not fall into one of these classes, the argument in the proof breaks down: we obtain that $|v_n| \equiv 1 \bmod 4$ or $|v_n| \equiv 2 \bmod 8$, which does not force the existence of a prime divisor $p \equiv 3 \bmod 4$. It is still possible, of course, that such prime divisors exist. We will see below that in some cases, we can show that v_n (or a related number) splits into a product whose factors can be shown to have suitable prime divisors, but this results in a much thinner set (parameterized by a quadratic expression) than what Lemma 4.3 gives.

5 The Klein four-group of characters

Lemma 4.3 tells us that for half of the rational $a \in \mathcal{A}$, we can show that ρ_a is surjective, *if* we can also produce ϕ_0 and ϕ_1. Recall that this is equivalent to a having two out of ϕ_0, ϕ_1, and τ: together with the trivial character, the corresponding three characters form a Klein four-group inside the character group of $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$.

We will now collect sufficient conditions for each of these to be the case. The proofs are rather straight-forward, so we keep them concise; for details, see the Lean formalization at [Sto26, Part I of the blueprint].

We note that for a quadratic character ψ on $\mathbb{Q}^\times$, we have

$$\psi(c_1) = \psi(-rs), \qquad \psi(c_2) = \psi\big(r(r+s)\big) \quad \text{and} \quad \psi(c_1c_2) = \psi\big(-s(r+s)\big).$$

Proposition 5.1 (Criteria for ϕ_0). *a has ϕ_0 if one of the following holds.*

(Z1) *$|r|$ is not a square.*

(Z2) *A prime $p \equiv \pm 3 \bmod 8$ divides $r + 2s$.*

(Z3) *A prime $p \equiv 5 \bmod 8$ divides $r^2 + s^2$ with $\chi_p\big((r-s)s\big) = -1$.*

(Z4) *A prime* $p \equiv 1 \bmod 4$ *divides* r *with* $\chi_p(-rs) = -1$*; or* $8 \mid r$ *and* $\chi_8(-rs) = -1$.

Proof. (Z1) $v_p(c_n) = v_p(r)$ for all n, so val_p is a suitable character. (Z2) $c_n \equiv 2 \bmod p$ for all n, so we can take χ_p. (Z3) Here $a^2 \equiv -1 \bmod p$, so the sequence $(c_n)_{n\geq 1}$ reduces mod p to $(-a, a-1, -a, a-1, \ldots)$; moreover, $\chi_p(-a) = \chi_p(a) = (-1)^{(p-1)/4} = -1$ and $\chi_p(a-1) = \chi_p((r-s)s) = -1$. So we can take χ_p. (Z4) We see that $\chi_p(c_1) = \chi_p(-1)\chi_p(rs)$ and $\chi_p(c_n) = \chi_p(rs)$ for $n \geq 2$ (and similarly for χ_8 in the second case). So we can take χ_p, respectively, χ_8. □

Proposition 5.2 (Criteria for ϕ_1)**.** a *has* ϕ_1 *if one of the following holds.*

(O1) $a > 0$.

(O2) $a \equiv 1 \bmod 4$*, i.e.,* s *is odd and* $r \equiv s \bmod 4$.

(O3) *For some prime* p *dividing* s*,* $-rs$ *is not a square in* $\mathbb{Q}_p$*, and if* $p = 2$ *and* $v_2(s) = 2$*, then moreover* $\chi_4(-rs) = -1$.

(O4) *A prime* $p \equiv 3 \bmod 4$ *divides* r *with* $\chi_p(-rs) = -1$*; or* $4 \mid r$ *and* $\chi_4(-rs) = -1$.

In particular, by (O3)*,* a *has* ϕ_1 *if* s *is not a square.*

Proof. (O1) The sign character does it. (O2) We can take χ_4 or $\chi_4 \cdot \mathrm{val}_2$ (consideration mod 8). Note that $\mathrm{val}_2(c_n) = (-1)^{n-1}$. (O3) Note that $rw_n \equiv r^{2^{n-1}} \bmod s$ for $n \geq 2$. If $v_p(s)$ is odd, we can take val_p. If $v_p(s) \geq 2$ is even and p is odd, we can take χ_p (the assumption on $-rs$ implies that $\chi_p(c_1) = -1$; all c_n for $n \geq 2$ are p-adic squares). If $v_2(s) \geq 2$ is even, then $rw_n \equiv 1 \bmod 8$ for $n \geq 3$ and $rw_2 \equiv 1 + rs \bmod 8$. When $v_2(s) = 2$, the additional assumption implies that we can take χ_4. When $v_2(s) \geq 4$, we can take one of χ_4, χ_8 and $\chi_4 \cdot \chi_8$. (O4) We can take χ_p, resp. χ_4; compare the proof of (Z4) above. □

Proposition 5.3 (Criteria for τ)**.** a *has* τ *if one of the following holds.*

(T1) $-1 < a < 0$.

(T2) *A prime* $p \equiv 3 \bmod 4$ *divides* r *with* $\chi_p(-rs) = 1$*; or* $4 \mid r$ *and* $\chi_4(-rs) = 1$.

(T3) *A prime* $p \equiv 3 \bmod 4$ *divides* $r + s$ *with* $\chi_p\big(-s(r+s)\big) = -1$*; or* $4 \mid r + s$ *and* $\chi_4\big(-s(r+s)\big) = -1$*; or* $|r + s|$ *is not a square and divisible by* 4 *or by a prime* $p \equiv 3 \bmod 4$.

(T4) $\alpha' < a < -1$ *(that is,* $c_3 < 0 < c_4$*), and* $|r + s|$ *is not a square.*

Proof. (T1) Take the sign character. (T2) We can take χ_p resp. χ_4; compare the proof of (Z4) above. (T3) Let $\chi = \chi_p$, resp. χ_4. Here $a \equiv -1 \bmod p$, so modulo p, $(c_n)_{n\geq 1}$ reduces to $(1, 0, -1, 0, -1, \ldots)$. We also have $(c_{2n})_p \equiv (c_2)_p \bmod p$ for all $n \geq 1$; this follows from $c_{2k} \equiv c_2 \bmod c_2^2$. So χ is constant on (c_{2n}). We can then take χ when $\chi(c_2) = -1$ and $\chi \cdot \mathrm{val}_q$ for a prime q with $v_q(r+s)$ odd otherwise. (T4) Take $\mathrm{sign} \cdot \mathrm{val}_q$ with $v_q(r+s)$ odd. □

We will say that "a satisfies (Z)", "a satisfies (O)", "a satisfies (T)" when a satisfies one of the conditions (Z1)–(Z4), one of the conditions (O1)–(O4), and one of the conditions (T1)–(T4), respectively.

We note a useful consequence of (Z1), (Z2) and (O2).

Lemma 5.4. *If* $a \equiv 1 \bmod 4$*, then* a *has* ϕ_0 *and* ϕ_1.

Proof. We have that s is odd and $r \equiv s \bmod 4$. By assumption, (O2) holds. If $|r|$ is not a square, then (Z1) holds. Otherwise, $r = \pm t^2$ with t odd, so $r \equiv \pm 1 \bmod 8$ and $s \equiv \pm 1 \bmod 4$ (with the same sign); then $r + 2s \equiv \pm 3 \bmod 8$, so (Z2) holds. □

One can extend these criteria to cover some further cases. See the Lean formalization at [Sto26, Blueprint, Section B.1 of Part I] for details.

6 The main result

As before, we write $a = r/s$ in lowest terms.

Combining the various pieces, we obtain the following result.

Theorem 6.1. *Assume that $a \in \mathcal{A}$ and (r, s) satisfies one of the conditions (C1)–(C3) in Lemma 4.3. Then ρ_a is surjective if and only if a has ϕ_0 and ϕ_1. Moreover, the following hold.*

(1) *If $a > \alpha'$ (so $\varepsilon_a = 1$) or $a \le -2$ (so $\varepsilon_a = -1$), then the following are equivalent.*
   (i) *ρ_a is surjective;*
   (ii) *(c_1, c_2) is 2-independent;*
   (iii) *none of $-rs$, $r(r+s)$, $-s(r+s)$ is a square.*

(2) *If $\alpha < a < \alpha'$ (so $\varepsilon_a = 1$), then ρ_a is surjective in each of the following cases.*
   (a) $(r, s) \equiv (1, 1)$, $(1, 3)$ *or* $(3, 2) \bmod 4$.
   (b) $(r, s) \equiv (2, 1) \bmod 4$ *and* ($r \equiv 2 \bmod 8$ *or* $s \equiv 5 \bmod 8$).
   (c) $(r, s) \equiv (0, 3) \bmod 4$ *and* ($|r|$ *is not a square in* $\mathbb{Q}_2$ *or* $r + 2s \equiv \pm 6 \bmod 16$).
   (d) $(r, s) \equiv (3, 0) \bmod 4$, $r \equiv 3 \bmod 8$ *and* $s \not\equiv 4 \bmod 32$.
   (e) *a satisfies at least two of* (Z), (O) *and* (T).

Proof. If ρ_a is surjective, then a has ϕ_0 and ϕ_1 by Proposition 2.3. Conversely, if a has ϕ_0 and ϕ_1, then a has ϕ_n for all $n \ge 2$ by Lemma 4.3, so ρ_a is surjective by Proposition 2.3.

In (1), the equivalence of (ii) and (iii) and the implication "(i) $\Rightarrow$ (ii)" are clear. So it suffices to show that a has ϕ_0 and ϕ_1 under (iii) in (1) and in each of the cases in (2). Since this is implied by having two out of ϕ_0, ϕ_1 and τ, case (2e) is done; in the other cases, we show that condition (2e) is also satisfied. Recall that (Z1) holds if $|r|$ is not a square, (O3) if s is not a square, and (T3) if $|r + s|$ is not a square and $|r + s| \equiv 0$ or $3 \bmod 4$. Also, when $a < -1$, so that $r < 0$ and $r + s < 0$, (iii) implies that at most one of $|r|$, s and $|r + s|$ is a square, since the products of two of them are then $-rs$, $r(r + s)$ and $-s(r + s)$.

(1)
- $a > 0$. (O1) holds, and r and $r + s$ are positive and not both squares (since $r(r + s)$ is not a square by assumption). In all cases except $(r, s) \equiv (1, 1) \bmod 4$, this gives (Z1) or (T3). In the case $(r, s) \equiv (1, 1) \bmod 4$, we have $a \equiv 1 \bmod 4$, and Lemma 5.4 applies.
- $-1 < a < 0$. (T1) holds, and $|r|$ or s is not a square, giving (Z1) or (O3).
- $\alpha' < a < -1$. At most one of $|r|$, s and $|r + s|$ is a square; we obtain at least two of (Z1), (O3) and (T4).
- $a \le -2$. At most one of $|r|$, s and $|r + s|$ is a square. In all classes except $(r, s) \equiv (3, 3)$, we obtain at least two of (Z1), (O3) and (T3). When $(r, s) \equiv (3, 3) \bmod 4$, we have $a \equiv 1 \bmod 4$, and Lemma 5.4 applies.

(2) In each of the cases (a)–(d), one can show that two of (Z), (O) and (T) hold. First note that $r < 0$ and $r + s < 0$.

(a) $(1,1)$: (Z1) and (O2). $(1,3)$: (Z1) and (O3). $(3,2)$: (O3) and (T3).

(b) $v_2(r) = 1$ gives (Z1). If $s \equiv 5 \bmod 8$, then (O3) holds. If $r \equiv 2 \bmod 8$, then a prime $p \equiv 3 \bmod 4$ divides $|r|$, so (O4) or (T2) holds.

(c) $s \equiv 3 \bmod 4$ gives (O3). If $|r|$ is not a 2-adic square, then it is not a square in $\mathbb{Z}$ and we have (Z1). If $r + 2s \equiv \pm 6 \bmod 16$, then we obtain (Z2).

(d) $|r| \equiv 5 \bmod 8$ gives (Z1). Here we have $v_2(s) \geq 2$, and if $v_2(s) = 2$, then $s_2 \not\equiv 1 \bmod 8$ by assumption. This gives (O3), with $p = 2$ unless $v_2(s)$ is even and $s_2 \equiv 5 \bmod 8$; then s is not a square, so (O3) holds with an odd prime p. □

Since we obtain ϕ_0 when $|r|$ is not a square and ϕ_1 when s is not a square (from val_p with a prime p dividing r, resp. s, with odd multiplicity), we see that ρ_a is surjective outside a subset of relative density zero within the set of $a \in \mathbb{Q}$ such that $\alpha < a < \alpha'$ and a satisfies one of (C1)–(C3). Even in cases where the conditions (Z), (O) and (T) are not sufficient to conclude that a has ϕ_0 and ϕ_1, we can try to find suitable characters by considering χ_p for various p and their products. In all cases we checked, either (c_1, c_2) is 2-dependent, or else such characters can be found. There are also some further sufficient conditions that apply in certain subfamilies. We refer to the formalization [Sto26, Blueprint, Appendix B of Part I] for their statements and proofs. The computational observations mentioned above suggest the following strengthening of Theorem 6.1.

Conjecture 6.2. *Let $a \in \mathcal{A}$ satisfy one of (C1)–(C3). Then ρ_a is surjective as soon as ρ_a^2 is.*

By Proposition 1.7, ρ_a^2 is surjective if and only if (c_1, c_2) is 2-independent. So the conjecture holds when $a > \alpha'$ or $a \leq -2$, by Theorem 6.1(1), and for $\alpha < a < \alpha'$, it is equivalent, by the first statement of Theorem 6.1, to the statement that a has ϕ_0 and ϕ_1 whenever (c_1, c_2) is 2-independent.

Figure 1 visualizes which rational numbers a (represented as pairs (s, r) in the plane) have ρ_a surjective and are covered by Theorem 6.1 (full circles) and Theorem 7.3 from the next section (full squares). The hollow circles mark cases where ρ_a is not surjective: the colored ones fall under Theorem 6.1; the black ones indicate numbers a such that the first six c_n (in fact, it is the first four in all cases in the plot) are 2-dependent. Gray dots represent values that are unresolved. The shaded gray wedge corresponds to the interval $(-2, \alpha)$; our methods do not cover this interval. Finally, the shaded light yellow wedge corresponds to the interval (α, α'), where the congruence conditions (C1)–(C3) are not always (known to be) sufficient for the implication "(c_1, c_2) 2-independent $\Rightarrow$ ρ_a surjective" to hold.

We deduce the main results of [Sto92] and [Li20].

Corollary 6.3. *Let a be an integer such that one of the following holds.*

(1) $a > 0$ *and* $a \equiv 1$ *or* $2 \bmod 4$.

(2) $a < 0$, $a \equiv 0 \bmod 4$ *and* $-a$ *is not a square.*

(3) $a < 0$, $a \equiv 3 \bmod 4$ *and neither of* $-a$ *and* $-a - 1$ *is a square.*

Then ρ_a is surjective.

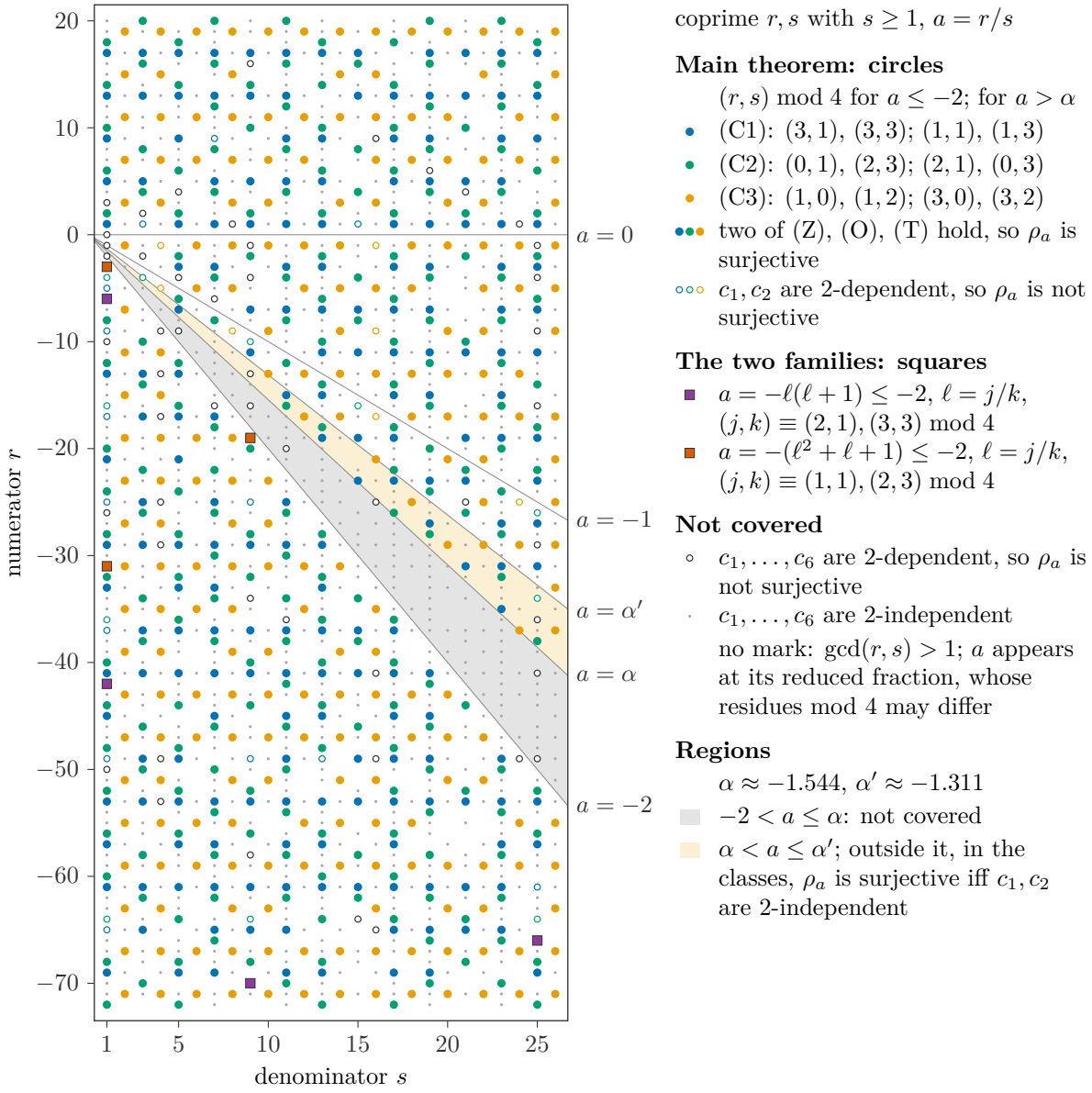


FIGURE 1. Coverage of Theorems 6.1 and 7.3.

Proof. Since $a = -1$ is excluded by the condition "$-a$ is not a square", we have $a \leq -2$ or $0 < a$; in particular, $a \in \mathcal{A}$. Then, in each case, a (i.e., $(r, s) = (a, 1)$) satisfies one of (C1)–(C3), so part (1) of Theorem 6.1 applies.

In all cases, $c_2 = a(a+1)$ is a product of coprime nonzero integers differing by 1, hence a nonsquare. In the first case, $c_1 = -a$ and $c_2/c_1 = -(a+1)$ are negative, hence nonsquares. In the second case, $-(a+1) \equiv 3 \bmod 4$ is a nonsquare; the other conditions in the last two cases are explicitly assumed. □

7 Two quadratic families

The main work in the proof of Theorem 6.1, in a sense, is to show that a has ϕ_n for all $n \geq 2$. The vehicle for that is the existence of a prime $p \equiv 3 \bmod 4$ such that modulo p, the sequence (c_n) becomes stationary at index $n+1$. This follows when

$|v_n| \equiv 3 \bmod 4$ or $|v_n| \equiv 6 \bmod 8$. (We could use the character induced by the Jacobi symbol $\left(\frac{\cdot}{|v_n|}\right)$ or $\left(\frac{\cdot}{|v_n/2|}\right)$ to state this in a more "constructive" way, i.e., without relying on a factorization.) This is the reason for the conditions (C1)–(C3): they provide the required congruences; in all other cases the congruences fail. This is also responsible for the fact that we only obtain a result for a set of relative density $1/2$ in $\mathbb{Q} \setminus (-2, \alpha)$.

In some cases, however, we can get a bit further. This is when we can show that $|v_n|$ (say; there is a variant of this argument using u_n as defined below), even though congruent to 2 mod 8, systematically factors as a product of two positive numbers one of which can be shown to be $\equiv 3 \bmod 4$ or $\equiv 6 \bmod 8$. The main result of this section, Theorem 7.3 below, exploits this idea, which is due to Li in [Li21], where essentially the special case for integral a was obtained.

We begin by giving additional criteria that provide us with ϕ_n in certain cases. We set, for $n \geq 1$,

$$u_n = r^2 w_n^2 - r w_n s^{2^{n-1}} + (r+s)s^{2^n-1};$$

then $c_{n+1} - c_n + 1 = u_n/s^{2^n}$ in lowest terms; in particular, u_n and s are coprime. Note that

$$c_{n+2} + c_n = (c_{n+1} + c_n)(c_{n+1} - c_n + 1),$$

so modulo a prime dividing u_n, we have $c_{n+2} \equiv -c_n$ (and then $c_{n+2k} \equiv -c_n$ for all $k \geq 1$ by induction).

Lemma 7.1. *Let $n \geq 1$ and let $p \equiv 3 \bmod 4$ be a prime with $p \mid u_n$ and $p \nmid r+s$. Then the values of χ_p on $(c_m)_{m \geq n}$ are $(x, y, -x, y, -x, \dots)$, with $x = \chi_p(c_n)$, $y = \chi_p(c_{n+1}) \in \{\pm 1\}$. Consequently, if a has ϕ_0 and some character alternates from n on, then a has ϕ_n.*

Proof. u_n is prime to s, so $\bar{c}_{n+2} = -\bar{c}_n$ in $\mathbb{Z}/p\mathbb{Z}$ by the considerations above; it follows that

$$(\bar{c}_m)_{m \geq n} = (\bar{c}_n, \bar{c}_{n+1}, -\bar{c}_n, \bar{c}_{n+1}, \dots).$$

These entries are units: $\bar{c}_n = 0$ would give $p \mid r w_n$, so $p \mid r+s$, and $\bar{c}_{n+1} = 0$ would give $\bar{c}_n = 1$ and then $\bar{a} = -1$, so again $p \mid r+s$. Finally, use that $\chi_p(-1) = -1$. The last statement then follows from Lemma 2.7, with $\chi = \chi_p$. □

Lemma 7.2. *Let $a \equiv 1 \bmod 4$ with $a \leq -2$. Then a has ϕ_n for every odd $n \geq 3$.*

Proof. Let $n \geq 3$ be odd. In $\mathbb{Z}/8\mathbb{Z}$, $\bar{c}_n = 5$ for $\bar{a} = 1$ and $\bar{c}_n = 1$ for $\bar{a} = 5$, and $s^{2^n} \equiv 1$, so $u_n \equiv \bar{c}_n^2 - \bar{c}_n + \bar{a} + 1 \equiv 6 \bmod 8$. Also $u_n > 0$: $u_n/s^{2^n} = c_n^2 - c_n + a + 1$, which is positive for $a \leq -2$ (as $c_n \geq -a \geq 2$ gives $c_n(c_n - 1) + a + 1 \geq (a+1)^2$). So a prime $p \equiv 3 \bmod 4$ divides u_n. It does not divide $r+s$: otherwise in $\mathbb{Z}/p\mathbb{Z}$ $\bar{a} = -1$ and $(\bar{c}_m)_{m \geq 1} = (1, 0, -1, 0, -1, \dots)$, so $\bar{c}_n^2 - \bar{c}_n + \bar{a} + 1 = 2 \neq 0$. Lemma 7.1, with the alternating character val_2 and with ϕ_0 from Lemma 5.4, gives ϕ_n. □

Theorem 7.3. *Let $\ell = j/k$ with coprime positive integers j and k.*

(1) *If $(j,k) \equiv (2,1)$ or $(3,3) \bmod 4$ and $a = -\ell(\ell+1)$ with $a \leq -2$, then ρ_a is surjective.*
(2) *If $(j,k) \equiv (1,1)$ or $(2,3) \bmod 4$ and $a = -(\ell^2 + \ell + 1)$ with $a \leq -2$, then ρ_a is surjective.*

Proof. Recall that it is sufficient to show that a satisfies two of the conditions (Z), (O) and (T) and that a has ϕ_n for all $n \geq 2$.

(1) Here $r = -j(j+k)$, $s = k^2$, $a < -2$, $\varepsilon_a = -1$, and $\nu_2(r) = 1$, so (Z1) holds. A prime $p \equiv 3 \bmod 4$ divides r (one of the positive integers $j+k$ and j is $\equiv 3 \bmod 4$), so (O4) or (T2) holds.

By Lemma 4.3 (1), a has ϕ_n (given that we know it has ϕ_0) when $n \geq 2$ and $|v_n|$ is divisible by a prime $p \equiv 3 \bmod 4$. Let $n \geq 2$, $W = -w_n$ and $V = k^{2^n-1}$. Then

$$-v_n = F_1F_2 \qquad \text{with} \qquad F_1 = (j+k)W - V \quad \text{and} \quad F_2 = jW + V\,.$$

Since $r < 0$ and $c_n = rw_n/k^{2^n} > 0$ (Lemma 3.1), we have $W > 0$, so $F_2 > 0$; then also $F_1 > 0$, as $F_1F_2 = -v_n > 0$. We have $s \equiv 1 \bmod 8$, so $w_{m+1} \equiv rw_m^2 + 1$, and as $r \equiv 2 \bmod 4$ and w_m is odd, $w_m \equiv r+1 \bmod 8$ for $m \geq 2$; thus $W \equiv j(j+k)-1$ and $V \equiv k \bmod 8$. This gives $F_1 \equiv 6 \bmod 8$ when $(j,k) \equiv (2,1) \bmod 4$ and $F_2 \equiv 6 \bmod 8$ when $(j,k) \equiv (3,3) \bmod 4$. In both cases, we obtain a prime $p \equiv 3 \bmod 4$ dividing $|v_n| = F_1F_2$.

(2) Here $r = -(j^2+jk+k^2)$, $s = k^2$, $r+s = -j(j+k)$, and $a \equiv 1 \bmod 4$, so (O2) holds. $|r| \equiv 3 \bmod 4$ is not a square, so (Z1) holds. Lemma 7.2 gives ϕ_n for odd $n \geq 3$; and val_2 alternates on c_n from $n = 1$.

Let $n \geq 2$ be even and $V = k^{2^n-1}$. Then $u_n = G_nH_n$ with

$$G_n = rw_n - (j+k)V = k^{2^n}(c_n - \ell - 1) \quad \text{and} \quad H_n = rw_n + jV = k^{2^n}(c_n + \ell)\,,$$

both positive as $c_n \geq -a = \ell^2 + \ell + 1$. By induction, we see that $w_n = (r+s)T$ with $T \equiv s^{2^{n-1}-2} \bmod (r+s)$, so j and $j+k$ divide w_n (as $r+s = -j(j+k)$). Replacing (j,k) by $(j+k,-k)$, that is, ℓ by $-1-\ell$, leaves a, and hence r, s, w_n and T, unchanged; it interchanges j and $j+k$ and changes the sign of V, so it interchanges G_n and H_n, and it maps the class $(2,3)$ to the class $(1,1) \bmod 4$. The following argument uses that $j > 0$, but not the sign of k; since $j+k > 0$, we can therefore assume that $(j,k) \equiv (1,1) \bmod 4$. We set

$$K = H_n/j = -r(j+k)T + V > 0\,.$$

Then:

- $K \equiv 3 \bmod 4$. In $\mathbb{Z}/4\mathbb{Z}$, $\bar{a} = 1$, so $\bar{c}_n = 2$ for even n, and $rw_n = s^{2^{n-1}}\bar{c}_n = 2$; also $V \equiv k \bmod 4$. So $jK = H_n \equiv 2 + jk$, hence $K \equiv 2 + k \equiv 3 \bmod 4$.
- K is prime to $r+s$. K is odd. Let q be an odd prime dividing $r+s = -j(j+k)$; then $q \nmid k$, and $r \equiv -k^2$ and $rT \equiv -k^{2^n-2} \bmod q$. So $K \equiv (j+k)k^{2^n-2} + V$, which is $2V$ if $q \mid j$ and V if $q \mid j+k$. The claim follows, as q does not divide $2V$.

So a prime $p \equiv 3 \bmod 4$ divides K, hence u_n, and $p \nmid r+s$. Lemma 7.1 with val_2 and ϕ_0 gives ϕ_n. □

Theorem 7.3 generalizes the results of [Li21]. The first case in the corollary below improves on [Li21, Thm. 3.3] (which covers even m); the second case is [Li21, Thm. 3.9].

Corollary 7.4. *If $m \in \mathbb{Z}_{\geq 0}$ and $a = -(4m+2)(4m+3)$ or $a = -((4m+1)(4m+2)+1)$, then ρ_a is surjective.*

Proof. Apply Theorem 7.3 with $k = 1$. □

8 Computational evidence

In this section, we report on computational evidence supporting the following conjecture.

Conjecture 8.1. *There is a natural number* N *such that for all* $a \in \mathbb{Q}$*, if* ρ_a^N *is surjective, then so is* ρ_a*.*

Forms of this conjecture are already present in the literature. Hindes [Hin15] considers the set $S^{(n)}$ of $a \in \mathbb{Q}$ such that ρ_a^{n-1} is surjective, but ρ_a^n is not. At the end of Section 3, he raises the question whether $S^{(n)} = \emptyset$ for all sufficiently large n, which is equivalent to Conjecture 8.1, and asks "Is it true for all $n \geq 5$?", which is Conjecture 8.1 with $N = 4$.

Our main result (Theorem 6.1) shows that Conjecture 8.1 holds for half of the rational numbers outside the interval $(-2, \alpha')$ with $N = 2$ (and within the interval (α, α') for half of the numbers outside at most a set of density zero). This rests on the fact that such a have ϕ_n for all $n \geq 2$ (assuming they have ϕ_0), which follows from the fact that $a \in \mathcal{A}$ and satisfies one of the congruence conditions (C1)–(C3). Our method cannot reach much beyond this (except for the thin quadratic families discussed in Section 7).

So, to investigate whether Conjecture 8.1 has a chance of being true, we tested all rational numbers a of height up to 10^6 (i.e., $a = r/s$ with $|r| \leq 10^6$ and $1 \leq s \leq 10^6$) for quadratic relations among $c_1, \ldots, c_{20}$, i.e., we determined the non-empty subsets T of $\{1, \ldots, 20\}$ with $\max T \geq 4$ such that $\prod_{n \in T} c_n$ is a square in $\mathbb{Q}$. This gives the following result.

Proposition 8.2. *Let* $a \in \mathbb{Q}$ *be of height at most* 10^6*. Assume that* ρ_a^3 *is surjective and* $a \notin \{2/3, -6/7\}$*. Then* ρ_a^{20} *is surjective. For* $a \in \{2/3, -6/7\}$*,* ρ_a^3 *is surjective, but* ρ_a^4 *is not.*

Proof. Recall that by Proposition 1.7, ρ_a^n is surjective if and only if $(c_1, \ldots, c_n)$ is 2-independent.

Let $C_0 = 0 \in \mathbb{Z}[x]$ and $C_{n+1} = C_n^2 + x$ for $n \geq 0$ (so that for any given a, $c_1 = -C_1(a)$ and $c_n = C_n(a)$ for $n \geq 2$). There are monic polynomials $B_n \in \mathbb{Z}[x]$ for $n \geq 1$ such that $C_n = \prod_{d|n} B_d$ for all $n \geq 1$, and the pairwise resultants of the B_n are all ± 1; see [BEK20, Lemma 9.5], which the authors attribute to Bjorn Poonen. (The polynomials C_n are known as Gleason polynomials; the complex roots of the B_n are the centers of period exactly n in the Mandelbrot set.)

By Remark 1.3, the only rational roots of any of the C_n and therefore of any of the B_n are 0 and -1, for which already ρ_a^1 is not surjective, so that the claim holds for them. We also know that ρ_{-2}^2 is not surjective ($c_1 c_2 = 2^2$ in this case). We can therefore assume from now on that $a \notin \{-2, -1, 0\}$. So let $\mathcal{H}$ denote the set of all rational numbers a of height at most 10^6 except $a = -2$, $a = -1$, and $a = 0$. Since no factor in a product $\prod_{n \in T} c_n$ vanishes, the quadratic relations among the c_n correspond one-to-one to quadratic relations between the numbers $B_n(a)$, except for a sign coming from the fact that $c_1 = -C_1(a)$, which is therefore present if and only if the relation of the c_n involves c_1; this is equivalent to the degree of the product of B_n obtained by canceling square factors from the corresponding product of the C_n being odd (note that $\deg C_n = 2^{n-1}$ is even for $n \geq 2$). It is not hard to show by induction that $B_n(-1) = B_n(0) = 1$ and $B_n(-2) = (-1)^{\deg B_n}$ for all $n \geq 3$. This implies that even-degree products of the B_n

with $n \geq 3$ evaluated at a are 3-adic squares for all $a \in \mathbb{Q}$; in particular, the negative of such a product can never be a square in this case.

The fact that the pairwise resultants of the B_n are units implies that in any product of values $B_n(a)$ (for $a \in \mathbb{Q}$ and pairwise distinct indices n) that is a square up to a sign, each value of a subproduct such that the corresponding product of the B_n has even degree must be a square up to a sign. This follows from considering the even-degree homogenization of the product polynomial evaluated at the numerator and denominator of a: the unit resultants imply coprimality of the factors coming from the various B_n (and, as the B_n are monic, with the denominator as well). When the degree of the product polynomial is odd, this argument also shows that the value of each odd-degree factor must be a square up to sign. The 3-adic consideration above then lets us exclude the negative sign in each case where we have an even-degree product not involving B_1 or B_2. Also note that the degree of B_n is odd if and only if n is squarefree. From these considerations we deduce that the claim follows if we can verify the following.

(1) For $8 \leq n \leq 20$ not squarefree and $a \in \mathcal{H}$, $B_n(a)$ is not a square.
(2) For $a \in \mathcal{H}$, $B_4(a)$ is a square if and only if $a = 2/3$ or $a = -6/7$.
(3) For m, n squarefree with $3 \leq m < n \leq 19$ and $a \in \mathcal{H}$, $B_m(a)B_n(a)$ is not a square.
(4) For $5 \leq n \leq 19$ squarefree and $a \in \mathcal{H}$, both $B_1(a)B_n(a)$ and $B_2(a)B_n(a)$ are not of the form $\pm\square$.
(5) For $5 \leq n \leq 19$ squarefree and $a \in \mathcal{H}$, $B_n(a)$ is not of the form $\pm\square$.

(Note that these imply that for $a \in \mathcal{H}$, there cannot be any relation $\prod_{n\in T} c_n = \square$ with $5 \leq \max T \leq 20$, and there is such a relation with $\max T = 4$ only for $a = 2/3$ and $a = -6/7$.)

Each of these claims is equivalent to saying that on one or several hyperelliptic curves, all finite rational points whose x-coordinate has height at most 10^6 have x-coordinate in $\{-2, -1, 0\}$ (or in $\{-2, -6/7, -1, 0, 2/3\}$ in the case $n = 4$). The `ratpoints` program [ratpoints] (written and maintained by the author) can search for rational points on hyperelliptic curves very efficiently. With some patches to enable it to deal with polynomials of large degrees ($\deg B_{20} = 523\,770$) and with large coefficients, we ran it on all these curves with a height bound of 10^6. This took about three hours; the most time-intensive part was to generate the more than 170 GB of coefficient data for the polynomials. The results confirm that the only rational points up to that height on these curves are the expected ones.

For $a = 2/3$ and $-6/7$, we easily check that (c_1, c_2, c_3) is 2-independent, whereas c_2c_4 is a square in both cases (this is equivalent to $B_4(a)$ being a square). This implies the claims in the last sentence of the statement of Proposition 8.2. □

Remark 8.3. Consider $\prod_{n\in T} B_n = (-1)^t y^2$, where $t = \sum_{n\in T} \deg B_n$ is the degree of the left-hand side. (Note that here T denotes a set of indices in a product of the B_n, rather than in a product of the c_n as in the proof above. The sign $(-1)^t$ is negative if and only if the corresponding product of C_n's involves C_1; recall that $C_1(a) = a = -c_1$.) For $T = \{1\}, \{2\}, \{1, 2\}$, the resulting equation gives a curve of genus 0 whose rational points are easily parameterized. As $\deg B_3 = 3$, we obtain curves of genus 1 for $T = \{3\}, \{1, 3\}$, and $\{2, 3\}$. The first and the third of these are elliptic curves of rank 1 and therefore lead to infinitely many rational points, whereas the second one has rank 0 and rational points only at infinity and with $x = 0$. For $T = \{1, 2, 3\}$ we obtain a genus 2 curve whose Jacobian has a finite group of rational points, and we easily obtain that its rational

points are at infinity or have $x = 0$ or -1. For these results, see also [Hin13]. For $T = \{4\}$ we obtain another genus 2 curve. Its Jacobian has Mordell–Weil rank 3, so Chabauty's method does not apply, and the Galois group of B_4 is maximal (i.e., S_6), so alternative ways of determining the rational points are not very promising; see also [Hin15], where also a computation is described that shows that there are no further rational points up to height 10^{100} by running a Mordell–Weil sieve. All other curves have genus at least 3; Hindes in [Hin18, Lemmas 4.2, 4.3] uses the Chabauty–Coleman method to show that there are no further rational points on $y^2 = \pm B_5(x)$ (these are curves of genus 7 whose Jacobians have Mordell–Weil ranks 5 and 3). In the preprint [Sto19], we show that the curve $y^2 = C_7(x) = B_1(x)B_7(x)$ of genus 31 has only the points at infinity and $(0, 0)$ as rational points (assuming GRH for the computation of the class group of the number field of degree 63 obtained by adjoining a root of B_7 to $\mathbb{Q}$). We also show that the curves $y^2 = \pm C_5(x) = \pm B_1(x)B_5(x)$ of genus 7 have no unexpected rational points.

Remark 8.4. Restricting the parameter a to integer values, one can show a bit more. Hindes in [Hin18, Thm. 1.3] shows that in this case, the surjectivity of ρ_a^3 implies that of ρ_a^5 and conjectures that this also implies that ρ_a is surjective (Conj. 1.5 in *loc. cit.*). This is Conjecture 8.1 for integral a with $N = 3$. In [Hin16], he shows that the integral version of the conjecture follows from the Vojta or the Hall–Lang conjecture.

Remark 8.5. One can consider a more general version of the question we are studying in this paper, by replacing 0 with another rational base point b and looking at the binary tree of iterated preimages of b under f_a. This leads to arboreal Galois representations $\rho_{a,b}^n$ and $\rho_{a,b}$. Equivalently, one considers the shifted polynomial $(x+b)^2+a-b$. In this setting, there is an example found by Jones [Jon12, Eq. (1)], where $\rho_{a,b}^8$ is surjective, but $\rho_{a,b}^9$ is not. (This example is x^2+1 with $b = -88\,255\,775\,491\,812\,351\,975\,604$.)

If we assume that no further (and thus fairly large) points exist on these curves, then the proof of Proposition 8.2 implies that if Conjecture 8.1 holds for some $N \le 20$, then it also holds for $N = 4$. This would suggest that the conjecture will likely indeed hold with $N = 4$. (The conjecture is wrong with $N < 4$, as the examples $a = 2/3$ and $-6/7$ show.)

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MATHEMATISCHES INSTITUT, UNIVERSITÄT BAYREUTH, 95440 BAYREUTH, GERMANY

Email address: Michael.Stoll@uni-bayreuth.de

URL: http://www.mathe2.uni-bayreuth.de/stoll/