ON THE L-FUNCTION OF THE CURVES $y^2 = x^5 + A$

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ABSTRACT. Let $C_A/\mathbb{Q}$ be the curve $y^2 = x^5 + A$, and let $L(s, J_A)$ denote the L-series of its Jacobian. Under the assumption that the sign in the functional equation for $L(s, J_A)$ is $+1$, we evaluate the critical value $L(1, J_A)$ in terms of the value of a theta series for $\mathbb{Q}(\sqrt{5})$ depending on A at a CM point coming from $\mathbb{Q}(\zeta_5)$.

1. INTRODUCTION

In this note, we study the L-function of the genus two curve

$$C_A : y^2 = x^5 + A \tag{1.1}$$

and its arithmetic implication to the curve and its Jacobian J_A. Here A is a (non-zero) rational number. The arithmetic of C_A has been studied by the first author in [St1, St2] in a more general setting. Although elliptic curves have been extensively studied and a lot of progress has been made, little is known for higher genus curves or higher dimensional abelian varieties. The curves C_A are interesting, since their Jacobians J_A are not modular in the usual sense in that they are not isogenous to any quotient of $J_0(N)$, the Jacobian of a modular curve, for any integer N. (This is because J_A does not have real multiplication over $\mathbb{Q}$.) To put it another way, there is no classical (normalized) eigenform f of weight 2 such that

$$L(s, J_A) = \prod_{\sigma:\mathbb{Q}(f)\longrightarrow\mathbb{C}} L(s, f^\sigma)$$

where $\mathbb{Q}(f)$ is the subfield of $\mathbb{C}$ generated by the Fourier coefficients of f. So well-known results on the Birch and Swinnerton-Dyer conjecture (as for example Kolyvagin and Logachev [KL]) do not apply directly.

However, since J_A has complex multiplication by $\mathbb{Z}[\zeta_5]$ over $E = \mathbb{Q}(\zeta_5)$, there is a Hecke character η_A of E such that

$$L(s, J_A) = L(s, \eta_A)\,, \tag{1.2}$$

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compare [St2]. By results of Jacquet-Langlands [JL, Prop. 12.1], this implies that there is a Hilbert modular form h_A over $F = \mathbb{Q}(\sqrt{5})$ such that

$$L(s, J_A) = L(s, h_A)\,.$$

One then easily obtains that

$$L(s, J_A/F) = L(s, h_A)L(s, h_A^\tau)$$

where τ is the non-trivial automorphism of $F/\mathbb{Q}$; therefore J_A is modular in the usual sense over F.

In this note, we will give a method to express the central L-value $L(1, \eta_A)$ in terms of the value of a certain theta series for F, which depends on A, at a certain CM point coming from E (which may also depend on A). We carry this out explicitly in the special case $A = 1$, where the result is as follows (see Theorem 4.1).

$$L(1, J_1) = L(1, \eta_1) = \frac{\pi^2\,|I|^2}{5^{3/4}(1 - \cos(4\pi/5) - \sin(2\pi/5))}$$

with

$$I = \sum_{a,b\in\mathbb{Z}, a+b \text{ odd}} \exp\Big(-\frac{2\pi}{5}\big(\sin(\tfrac{2\pi}{5})^3(a+b\sqrt{5})^2 + \sin(\tfrac{4\pi}{5})^3(a-b\sqrt{5})^2\big)\Big)\,.$$

In particular, $L(1, J_1) \neq 0$. A 2-descent procedure as described in [St3] shows that $J_1(\mathbb{Q})$ is finite (of order 10). So the rank part of the Birch and Swinnerton-Dyer conjecture is verified in this case. See Section 6 at the end of this note for more comments on the Birch and Swinnerton-Dyer conjecture for the Jacobians J_A.

We mention that a similar explicit formula has been obtained in [RVY, Ya2, Ya3] for CM number fields E such that E has no roots of unity other than $\{\pm 1\}$ and that every prime of E above 2 is split in E/F. So the work presented here can be viewed as a complement. The main new ingredient is to work out what happens above 2 in the case considered here, where tenth roots of unity are present in E and 2 is inert, see the remark after Proposition 3.1 and Section 5.

This note is organized as follows. In Section 2, we introduce notations and collect necessary information about the character η_A. In Section 3, we use [Ya1] to obtain a general formula for the central L-value $L(1, \eta_A)$. In Section 4, we obtain the explicit formula for $L(1, \eta_1)$ alluded to above, leaving a local technical computation to Section 5.

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2. Preliminaries

Let $E = \mathbb{Q}(\zeta_5)$ and $F = \mathbb{Q}(\sqrt{5})$. We denote by $\mathcal{O} = \mathcal{O}_F$ the ring of integers of F. We fix an embedding of E into $\mathbb{C}$ by identifying $\zeta = \zeta_5$ with $e^{2\pi i/5}$. Let $\mathrm{N}\alpha$ denote the norm of $\alpha \in E$ from E to $\mathbb{Q}$. Set $\lambda = 1 - \zeta$ and denote by ρ the character of $\mathcal{O}_{E,\lambda}^\times$ of conductor λ^2 that is given by $\rho|_{\mathbb{Z}_5^\times} = 1$ and $\rho(\zeta) = \zeta^{-1}$. (This is χ_{λ^2} from [St2].)

Fix $\delta = \zeta^{-2} - \zeta^2 \in E$. Then $\Delta = \delta^2 \in F$. We also fix a CM type Φ of E as $\Phi = \{\sigma_2, \sigma_4\}$, where $\sigma_a(\zeta) = \zeta^a = e^{2\pi i a/5}$. Note that $\sigma_2(\delta) = 2i\sin(2\pi/5)$ and $\sigma_4(\delta) = 2i\sin(4\pi/5)$. In particular,

$$|\sigma_2(\delta)\sigma_4(\delta)| = \sqrt{5}\,. \tag{2.1}$$

In the following, $\left(\frac{\cdot}{\cdot}\right)$ denotes the usual Legendre symbol, whereas $\left(\frac{\cdot}{\cdot}\right)_n$ denotes the nth power residue symbol on E. We denote by $\mathbb{A} = \mathbb{A}_F$ the adele ring and by $\mathbb{A}^\times$ the idele group of F. Let $\chi_A = \eta_A|\cdot|_{\mathbb{A}}^{1/2}$ be the unitary counterpart of η_A. Then

$$L(\tfrac{1}{2}, \chi_A) = L(1, \eta_A)\,.$$

Lemma 2.1. *The restriction of χ_A to $\mathbb{A}^\times$ coincides with the quadratic Hecke character of F associated to E/F.*

Proof: By [St2, Cor. 3.2 and Lemma 3.6], the Größencharakter corresponding to χ_A is given (on elements α prime to $10A$) by

$$\tilde{\chi}_A(\alpha) = \left(\frac{A}{\alpha}\right)_2 \left(\frac{4A}{\alpha}\right)_5^4 \left(\frac{\alpha}{\lambda}\right)_2 \rho(\alpha)\sigma_2(\alpha)\sigma_4(\alpha)(\mathrm{N}\alpha)^{-1/2}\,.$$

If α is in F, then the first two power residue symbols take the value 1. The third symbol gives the quadratic Größencharakter belonging to E/F, ρ is trivial, and the last three factors cancel. $\square$

We will fix a 'canonical' additive character $\psi = \prod \psi_v$ of $\mathbb{A}/F$ as follows:

$$\psi_v(x) = e^{-2\pi i \lambda_v(x)}\,,$$

where

$$\lambda_v : F_v \xrightarrow{\mathrm{Tr}} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}\,,$$

if v is finite, and $\lambda_v(x) = -x$ if v is real. Let $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$. Note that the λ_v used here is the negative of that used in [La].

We need some information on χ_A. We will suppose that A is an integer not divisible by 5 or the tenth power of some prime number. Translating the results of [St2, Sect. 3] into the language used here, we obtain the following.

Proposition 2.2. *Let w be some place of E that lies over the place v of F. The 'root number' denotes Tate's local root number $\epsilon(\frac{1}{2}, \chi_{A,w}, \psi_{E_w}(\frac{1}{2}\,\cdot))$.*

(1) *If w is infinite (and complex), then $\chi_{A,w}(z) = |z|/z$. The root number is i, and $\chi_{A,w}(\delta) = -i$.*

(2) *If w is finite and inert in E/F, and w does not divide $10A$, then $\chi_{A,w}(\pi) = -1$ when $\pi \in F_v$ is a uniformizer, and $\chi_{A,w}$ is unramified. The root number and $\chi_{A,w}(\delta)$ are both 1.*

(3) *If $w \neq 2$ is finite and inert in E/F, and w divides A, then $\chi_{A,w}(\pi) = -1$ when $\pi \in F_v$ is a uniformizer. The conductor exponent of $\chi_{A,w}$ is 1, and on $\mathcal{O}_{E,w}^\times$, we have*

$$\chi_{A,w}(\alpha) = \left(\frac{\alpha}{\pi}\right)_2^e \left(\frac{\alpha}{\pi}\right)_5^e,$$

where e is the valuation of A at w (or at the prime p below w). The root number is $(-1)^{e+1}$ and $\chi_{A,w}(\delta) = (-1)^e \left(\frac{-1}{\mathrm{N}_{F/\mathbb{Q}}\pi}\right)^e$.

(4) *If $w = 2$, then $\chi_{A,2}(2) = -1$. The conductor exponent f_2 of $\chi_{A,2}$ is determined as follows. Write $A = 2^e B$ with B odd. Then*

$$f_2 = \begin{cases} 0 & \text{if } e = 8 \text{ and } B \equiv 1 \bmod 4, \\ 1 & \text{if } e < 8,\ e \text{ is even and } B \equiv 1 \bmod 4, \\ 2 & \text{if } e \text{ is even and } B \equiv -1 \bmod 4, \\ 3 & \text{if } e \text{ is odd.} \end{cases}$$

On $\mathcal{O}_{E,2}^\times$, we have

$$\chi_{A,2}(\alpha) = \left(\frac{2}{\mathrm{N}\alpha}\right)^e \left(\frac{-1}{\mathrm{N}\alpha}\right)^{(B-1)/2} \left(\frac{\alpha}{2}\right)_5^{e+2}.$$

The root number is $(-1)^{1+f_2+e}$ and $\chi_{A,2}(\delta) = (-1)^e$.

(5) *If $w = \lambda$, let $q(A) = (A^4 - 1)/5$. Then the conductor exponent f_λ of $\chi_{A,\lambda}$ is 1 if $5|q(A)$ and 2 otherwise. On $\mathcal{O}_{E,\lambda}^\times$, we have*

$$\chi_{A,\lambda}(\alpha) = \left(\frac{\alpha}{\lambda}\right)_2 \rho(\alpha)^{q(A)}.$$

Furthermore, $\chi_{A,\lambda}(\delta) = -\left(\frac{A}{5}\right)$, and the root number is -1 if $5|q(A)$ and $\left(\frac{Aq(A)}{5}\right)$ otherwise.

(6) *Let $N_A = 2^{f_2} \prod_{2\neq p|A} p$. Then the global root number of χ_A is given by $-\left(\frac{AN_A}{5}\right)$ if $5|q(A)$ and by $\left(\frac{q(A)N_A}{5}\right)$ otherwise.*

We will assume throughout this paper that the global root number of χ_A is 1. Then there is an $\alpha \in F^\times$ (unique up to norm from $E^\times$) such that for all places v of F,

$$\prod_{w|v} \epsilon(\tfrac{1}{2}, \chi_{A,w}, \psi_{E_w}(\tfrac{1}{2}\,\cdot))\chi_{A,w}(\delta) = \epsilon_v(\alpha)\,,$$

where the product is over places w of E above v, $\epsilon(\frac{1}{2}, \chi_{A,w}, \psi_{E_w}(\frac{1}{2}\,\cdot))$ are the local root numbers as in the preceding proposition, and ϵ_v is the local part of the Hecke character belonging to E/F. (This is automatically satisfied when v is split in E/F.) The following lemma follows easily from the preceding proposition.

Lemma 2.3. *Suppose the global root number of χ_A is 1. Then we can take*

$$\alpha = 2^{1+f_2} \prod_p p\,,$$

where the product is over primes $p \neq 2$ such that $p \mid A$, but leaving out the primes $p \equiv -1 \bmod 20$ with $v_p(A)$ odd. In particular, if A is a square such that $v_2(A) < 8$ then we can take

$$\alpha = \prod_{2 \neq p | A} p\,.$$

Proof: The first statement follows from the proposition at all places $v \neq \sqrt{5}$. For this last place, the claim then follows from the product formula and the fact that the global root number is 1. The second statement follows from the first, since $f_2 = 1$ and all $v_p(A)$ are even. □

3. A general formula

We will use the notation in [Ya1]. Let $S(\mathbb{A})$ be the space of Schwartz functions on the adeles $\mathbb{A}$ of F. Let $G = E^1$ be the norm one subgroup of $E^\times$, and view it as a unitary group of one variable. Associated to the data $(\chi_A, \psi, \delta, \alpha)$ chosen above, there is a Weil representation $\omega = \omega_{\alpha,\chi_A}$ of $G(\mathbb{A})$ on $S(\mathbb{A})$. For each character η of $G(\mathbb{A})$, there is an associated automorphic representation of $G(\mathbb{A})$ given by

$$\{\theta_\phi(\eta)(g) : \phi \in S(\mathbb{A}), g \in G(\mathbb{A})\}$$

Here $[G] = G(F)\backslash G(\mathbb{A})$ and

$$\theta_\phi(\eta)(g) = \int_{[G]} \sum_{x \in F} \omega_{\alpha,\chi_A}(gh)\phi(x)\,dh\,.$$

The representation is either η or zero. Whether it is η or not is related to the nonvanishing of the central L-value $L(\frac{1}{2}, \chi_A\tilde{\eta})$ where $\tilde{\eta}(z) = \eta(z/\bar{z})$. Here η has nothing do with η_A and will always be the trivial character 1 in this note. What we need is the following. Let $\phi = \prod_v \phi_v \in S(\mathbb{A})$ such that ϕ_v is the same as $\phi_{\bar{\eta}_v}$ (with η_v trivial) in [Ya1, Thm. 2.15] for $v \nmid 2$. We will choose ϕ_2 later on and let it be any locally constant function on F_2 with compact support for now. Then the proof of [Ya1, Thms 1.11 and 2.15] gives the following formula.

$$C_1 C_2 \frac{L(1,\eta_A)}{L(1,\epsilon)} = 2\,|\theta_\phi(1)(1)|^2\,.$$

Here

$$C_1 = \prod_{2\neq v|A,\ \text{inert}} (1+q_v^{-1})^{-1} \prod_{v|A,\ \text{split}} q_v^{-1}(1-q_v)^{-2}$$

is the constant in [Ya1, Thm. 2.15], and

$$C_2 = \int_{G_2} \langle \omega_{\alpha,\chi_A,2}(g)\phi_2, \phi_2\rangle dg\,, \tag{3.1}$$

where $G_2 = G(E_2) = E_2^1$. Moreover, $L(1,\epsilon) = \pi^2/25$.

Set $G_{2,1} = \{g \in G_2 :\ g \equiv 1 \bmod 2\}$. Then $G_2 = \langle\zeta\rangle \times G_{2,1}$. Set

$$U = G_{2,1} \prod_{v\neq 2,\ \text{nonsplit}} G_v \prod_{v\ \text{split}} \mathcal{O}_v^\times\,;$$

then $G_\mathbb{A} = G(F)U$ and $G(F)\cap U = \{\pm 1\}$. So

$$\begin{aligned}\theta_\phi(1)(1) &= \int_{[G]} \sum_{x\in F} \omega_{\alpha,\chi_A}(g)\phi(x)dg\\ &= \tfrac{1}{2}\int_U \sum_{x\in F} \omega_{\alpha,\chi_A}(g)\phi(x)dg\\ &= \tfrac{1}{2}\sum_{x\in F}\prod_v I_v(x)\,.\end{aligned}$$

Here

$$I_v(x) = \begin{cases} \int_{\mathcal{O}_v^\times} \omega_{\alpha,\chi_A,v}(g)\phi_v(x)dg & \text{if } v \text{ is split,}\\ \int_{G_v} \omega_{\alpha,\chi_A,v}(g)\phi_v(x)dg & \text{if } v\neq 2 \text{ is nonsplit,}\\ \int_{G_{2,1}} \omega_{\alpha,\chi_A,2}(g)\phi_2(x)dg & \text{if } v = 2.\end{cases}$$

By definition of ϕ_v, it is easy to see that $I_v(x) = \phi_v(x)$ when $v\nmid 2A$. So we have proved

Proposition 3.1. *Let the notation be as above. Then*

$$L(1,\eta_A) = \frac{\pi^2}{50C_1C_2}\Big|\sum_{x\in F}\prod_{v\nmid 2A}\phi_v(x)\prod_{v|2A} I_v(x)\Big|^2$$

when C_2 is nonzero.

There is always a choice of ϕ_2 making C_2 nonzero, e.g., we can take ϕ_2 to be a unitary eigenfunction of $(G_2, \omega_{\alpha,\chi_A,2})$; then $C_2 = 1$, compare [Ya1]. However, this eigenfunction is not explicitly constructed in the case $p = 2$. That is why the primes above 2 are assumed to be split in E/F in [RVY] and [Ya2, Ya3] as mentioned in the introduction. The fact that E has extra roots of unity also causes some technical problems. In Section 5, we will exhibit an explicit ϕ_2 that is almost an eigenfunction of $(G_2, \omega_{\alpha,\chi_A,2})$ and makes $C_2 \neq 0$. We refer to Section 5 for details and to Corollary 5.8 for the result.

Let $\mathrm{char}(X)$ denote the characteristic function of the set X.

Lemma 3.2.

$$\phi_v(x) = \begin{cases} \mathrm{char}(\mathcal{O}_v)(x) & \text{if } v \nmid 10\,A\,\infty \text{ and } \alpha \in \mathcal{O}_v^\times, \\ |2\sigma_j(\alpha\delta^3)|^{1/4} e^{-\pi|\sigma_j(\alpha\delta^3)|\sigma_j(x)^2} & \text{if } v = \sigma_j \in \{\sigma_2, \sigma_4\}. \end{cases}$$

Proof: The split case follows from [Ya1, Thm. 2.15]. The nonsplit case with $10A\alpha \in \mathcal{O}_v^\times$ follows from [Ya2, Prop. 1.2 and Cor. 1.4]. Finally, the infinite case follows from [Ya2, Lemma 1.1]. □

Note that $|\sigma_j(\alpha\delta^3)| = -i\,\sigma_j(-\alpha\delta^3)$, since α is totally positive and $\sigma_j(\delta)$ is purely imaginary with positive imaginary part. Hence

$$e^{-\pi|\sigma_j(\alpha\delta^3)|\sigma_j(x)^2} = e^{\pi i\sigma_j(-\alpha\delta^3)\sigma_j(x)^2},$$

and the expression under the absolute value in Prop. 3.1 is a theta function evaluated at the CM point given by $-\alpha\delta^3$.

4. The case $A = 1$

We now specialize to the case $A = 1$. By Lemma 2.3, we can then take any $\alpha \in \mathrm{N}_{E/F}E^\times$. We choose $\alpha = 1/5$ (which is a square in F). Then we have

$$\phi_{\sqrt{5}}(x) = 5^{1/4}\,\mathrm{char}(\mathcal{O}_{\sqrt{5}})(x)\,.$$

This follows from [Ya2, Prop. 1.2 and Cor. 1.4] again, since $\chi_{A,\lambda}$ has conductor (exponent) 1.

For $v = 2$, we will prove in Cor. 5.8 below that we can take $\phi_2 = \mathrm{char}(\frac{1}{2} + \mathcal{O}_2)$. The constant C_2 then has the value

$$C_2 = \frac{1 - \cos(\frac{4\pi}{5}) - \sin(\frac{2\pi}{5})}{5},$$

and $I_2(x) = \phi_2(x)$. Note that here $e = 0$ and $\left(\frac{-1}{\alpha}\right) = 1$.

Now, putting everything together (and using (2.1)), we finally get an explicit expression.

Theorem 4.1. *We have*

$$L(1,\eta_1) = C\,|I|^2 > 0$$

with

$$C = \frac{\pi^2}{5^{3/4}(1-\cos(4\pi/5)-\sin(2\pi/5))}$$

and

$$I = \sum_{x\in\frac{1}{2}+\mathcal{O}} \exp\Big(-\frac{\pi}{5}\big((2\sin(\tfrac{2\pi}{5}))^3\sigma_2(x)^2 + (2\sin(\tfrac{4\pi}{5}))^3\sigma_4(x)^2\big)\Big)\,.$$

5. Local computations at $v = 2$

In this section, we find a suitable ϕ_2 and compute related terms in the case that $f_2 = 1$ in the notation of Prop. 2.2.

We let A be arbitrary for now. To lighten notation, we drop the subscript 2. So $F = \mathbb{Q}_2(\sqrt{5})$ is the unramified extension of $\mathbb{Q}_2$ of degree 2 and $E = F(\delta) = \mathbb{Q}_2(\zeta)$ is the unramified extension of $\mathbb{Q}_2$ of degree 4. We also have $G = E_2^1$. Let $\mathcal{O} = \mathcal{O}_F = \mathbb{Z}_2[\frac{1+\sqrt{5}}{2}]$ be the ring of integers of F. Set for $k \ge 1$

$$\Gamma_k = \{g \in G : g \equiv 1 \bmod 2^k\}\,, \quad \Gamma'_k = \{g = x + y\delta \in G : y \in 2^k\mathcal{O}\}\,.$$

We also write simply χ instead of $\chi_{A,2}$.

The following two lemmas on the structure of G are easily verified.

Lemma 5.1. *The map $a \mapsto g_a = \frac{a+\delta}{a-\delta}$ is a bijection between $\mathbb{P}^1(F) = F \cup \{\infty\}$ and G. Moreover*

1. $g_a \in \Gamma_1$ *if and only if* $a^2 \not\equiv \Delta \bmod 4$.
2. $g_a \in \Gamma_1 - \Gamma'_2$ *if and only if* $a \in \mathcal{O}^\times$ *and* $a^2 \not\equiv \Delta \bmod 4$.
3. *For* $k > 1$, $g_a \in \Gamma'_k - \Gamma_k$ *if and only if* $a \in 2^{k-1}\mathcal{O}$.
4. *For* $k > 1$, $g_a \in \Gamma_k$ *if and only if* $1/a \in 2^{k-1}\mathcal{O}$.

Lemma 5.2.

1. $G = \langle\zeta\rangle \times \Gamma_1$.
2. *We have*

$$\zeta^{\pm1} = g_{\mp\frac{1+\sqrt{5}}{2}} = \tfrac{-1+\sqrt{5}}{4} \mp \tfrac{1+\sqrt{5}}{4}\delta\,,$$
$$\zeta^{\pm2} = g_{\mp\frac{3-\sqrt{5}}{2}} = -\tfrac{1+\sqrt{5}}{4} \mp \tfrac{1}{2}\delta\,.$$

3. *The group* Γ_1 *is generated by* Γ'_3, g_2, $g_{1+\sqrt{5}}$ *and* $g_{\frac{1-\sqrt{5}}{2}}$.

Recall that ([Ku, Prop. 4.8], see also [Ya1, (A1)])

$$\omega_{\alpha,\chi}(g)\phi = \mu_{\alpha,\chi}(g) r(\iota_\alpha(g))\phi\,,$$

where r is Rao's standard section [Rao],

$$\begin{aligned}\mu_{\alpha,\chi}(g) &= \chi(\delta(g-1))\gamma_F(\alpha y(1-x)\psi)(\Delta, -2y(1-x))_F \\ &= \chi(-\delta y(g-1))\gamma_F(\alpha y(1-x)\psi)\,,\end{aligned}$$

(note that $2(1-x)$ is the norm of $g-1$, so $(\Delta, 2(1-x))_F = 1$ and $(\Delta, -y)_F = \epsilon(-y) = \chi(-y)$) and

$$\iota_\alpha : G \longrightarrow \mathrm{Sp}(1) = \mathrm{SL}_2(F)\,, \quad g = x + y\delta \mapsto \begin{pmatrix} x & \Delta^2\alpha y \\ \frac{y}{\Delta\alpha} & x \end{pmatrix}.$$

Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Write

$$\iota_\alpha(g)w = \begin{pmatrix} x^{-1} & \Delta^2\alpha y \\ 0 & x \end{pmatrix} w \begin{pmatrix} 1 & -\frac{y}{\Delta\alpha x} \\ 0 & 1 \end{pmatrix} = m(x^{-1})n(\Delta^2\alpha x y)wn(-\frac{y}{\Delta\alpha x}) \tag{5.1}$$

for $g \in \Gamma_1$, and

$$\iota_\alpha(g) = \begin{pmatrix} \frac{\Delta\alpha}{y} & x \\ 0 & \frac{y}{\Delta\alpha} \end{pmatrix} w \begin{pmatrix} 1 & \frac{\Delta\alpha x}{y} \\ 0 & 1 \end{pmatrix} = m(\frac{\Delta\alpha}{y})n(\frac{xy}{\Delta\alpha})wn(\frac{\Delta\alpha x}{y}) \tag{5.2}$$

for $g \in G - \Gamma_1$. Here

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

Set

$$\xi_{\alpha,\chi}(g) = \begin{cases} \mu_{\alpha,\chi}(g)\gamma_F(2\Delta\alpha x y\psi) & \text{if } g = x + y\delta \in \Gamma_1, \\ \mu_{\alpha,\chi}(g) & \text{if } g \in G - \Gamma_1. \end{cases}$$

Then one has by (5.1) and [Rao, Thm. 3.5, Cor. 4.3]

$$\omega_{\alpha,\chi}(g)\phi(u) = \xi_{\alpha,\chi}(g) r(\iota_\alpha(g)w)\hat{\phi}(-u) \tag{5.3}$$

for every $\phi \in S(F)$ and $g \in \Gamma_1$, where

$$\hat{\phi}(u) = r(w)\phi(u) = \int_F \phi(v)\psi(-uv)\,dv$$

is the Fourier transform of ϕ.

From now on, we will assume that $f_2 = 1$ in the notation of Prop. 2.2. Then we can take $\alpha \in \mathcal{O}^\times$ to be a unit. We have $\chi(2) = -1$, and on $\mathcal{O}_E^\times$, χ is a character of order 5 which is trivial on $1 + 2\mathcal{O}_E$. Furthermore, $\chi(\delta) = 1$ and χ is trivial on $\mathcal{O}^\times$.

Lemma 5.3. *Assume $\alpha \in \mathcal{O}^\times$. For $\beta \in \frac{1}{2}\mathcal{O}$, set $f_\beta = \mathrm{char}(\beta + \mathcal{O})$. Then*

$$\omega_{\alpha,\chi}(g) f_\beta = \xi_{\alpha,\chi}(g) \psi(\frac{\Delta^2 \alpha y}{2x} \beta^2) f_\beta$$

for all $g = x + y\delta \in \Gamma_1$.

PROOF: We have the following explicit formulas (cf. [Rao, Thm. 3.5]).

$$\begin{aligned} r(n(b))\phi(u) &= \psi(\tfrac{1}{2} b u^2)\phi(u) \\ r(m(a))\phi(u) &= |a|^{1/2}\phi(au) \\ r(w)\phi(u) &= \hat{\phi}(u) \\ r(m(a)n(b)wm(a')n(b')) &= r(m(a))r(n(b))r(w)r(m(a'))r(n(b')) \end{aligned}$$

We first have to compute $\hat{f}_\beta$.

$$\begin{aligned} \hat{f}_\beta(u) &= \int_F f_\beta(v)\psi(-uv)\, dv \\ &= \int_{\mathcal{O}} \psi(-u(\beta + v))\, dv \\ &= \psi(-u\beta)\, \mathrm{char}(\mathcal{O})(u) \end{aligned}$$

(use that $\widehat{\mathrm{char}(\mathcal{O})} = \mathrm{char}(\mathcal{O})$). Now we use (5.1) and (5.3) together with above explicit formulas and the fact that $x \in 1 + 2\mathcal{O}$ and $y \in 2\mathcal{O}$ when $g \in \Gamma_1$. This gives

$$\begin{aligned} \omega_{\alpha,\chi}(g) f_\beta(u) &= \xi_{\alpha,\chi}(g) r(\iota_\alpha(g)w) r(-w) f_\beta(u) \\ &= \xi_{\alpha,\chi}(g) r(m(x^{-1}) n(\Delta^2 \alpha x y) w n(-\tfrac{y}{\Delta\alpha x})) \psi(\beta u)\, \mathrm{char}(\mathcal{O})(u) \\ &= \xi_{\alpha,\chi}(g) r(m(x^{-1}) n(\Delta^2 \alpha x y) w) \psi(-\tfrac{y}{2\Delta\alpha x} u^2) \psi(\beta u)\, \mathrm{char}(\mathcal{O})(u) \end{aligned}$$

(use that $-\frac{y}{2\Delta\alpha x} u^2 \in \mathcal{O}$)

$$\begin{aligned} &= \xi_{\alpha,\chi}(g) r(m(x^{-1}) n(\Delta^2 \alpha x y) w) \psi(\beta u)\, \mathrm{char}(\mathcal{O})(u) \\ &= \xi_{\alpha,\chi}(g) r(m(x^{-1}) n(\Delta^2 \alpha x y)) \int_{\mathcal{O}} \psi(\beta v)\psi(-uv)\, dv \\ &= \xi_{\alpha,\chi}(g) r(m(x^{-1}) n(\Delta^2 \alpha x y))\, \mathrm{char}(\mathcal{O})(u - \beta) \\ &= \xi_{\alpha,\chi}(g) r(m(x^{-1}) n(\Delta^2 \alpha x y))\, \mathrm{char}(\beta + \mathcal{O})(u) \\ &= \xi_{\alpha,\chi}(g) r(m(x^{-1})) \psi(\tfrac{1}{2}\Delta^2 \alpha x y u^2)\, \mathrm{char}(\beta + \mathcal{O})(u) \\ &= \xi_{\alpha,\chi}(g) |x|^{-1/2} \psi(\tfrac{\Delta^2 \alpha y}{2x} u^2)\, \mathrm{char}(\beta + \mathcal{O})(x^{-1} u) \end{aligned}$$

(use that $x \in 1 + 2\mathcal{O}$ and $\beta \in \frac{1}{2}\mathcal{O}$)

$$= \xi_{\alpha,\chi}(g)\psi(\tfrac{\Delta^2\alpha y}{2x}u^2)\,\mathrm{char}(\beta + \mathcal{O})(u)$$

(use that $u \in \beta + \mathcal{O}$ and that $\frac{\Delta^2\alpha y}{2x} \in \mathcal{O}$)

$$= \xi_{\alpha,\chi}(g)\psi(\tfrac{\Delta^2\alpha y}{2x}\beta^2)f_\beta(u)\,.$$

□

Lemma 5.4. *Let $\alpha \in \mathcal{O}^\times$ and let $\varphi(g) = \xi_{\alpha,\chi}(g)$. Then φ is a character on Γ_1 that does not depend on α. It is trivial on the subgroup generated by Γ_3', $g_{1+\sqrt{5}}$ and $g_{\frac{1-\sqrt{5}}{2}}$, and $\varphi(g_2) = -1$.*

Proof: Write $g_a = x + y\delta$, then

$$x = \frac{a^2 + \Delta}{a^2 - \Delta}\,, \quad y = \frac{2a}{a^2 - \Delta}\,, \quad 1 - x = -\frac{2\Delta}{a^2 - \Delta}\,.$$

We will drop the subscript F on γ and the Hilbert norm residue symbol. We have

$$\begin{aligned}\gamma(\alpha y(1-x)\psi)\gamma(2\Delta\alpha xy\psi) &= \gamma(2\Delta x(1-x),\psi)\gamma(\psi)^2(\alpha y(1-x), 2\Delta\alpha xy)\\ &= \gamma(-(a^2+\Delta),\psi)\gamma(-1,\psi)(-\alpha a\Delta, \alpha a\Delta(a^2+\Delta))\\ &= \gamma(a^2+\Delta,\psi)(\alpha a\Delta, a^2+\Delta)\,.\end{aligned}$$

Here we have used the fact that $\gamma(\psi)^2 = \gamma(-1,\psi)^{-1} = -1 = \gamma(-1,\psi)$. Now, for $g_a \in \Gamma_1$ and $\alpha \in \mathcal{O}^\times$, we have $(\alpha, a^2 + \Delta) = 1$ (since $a^2 + \Delta$ is a square times something that is a square mod 4). So

$$\varphi(g) = \xi_{\alpha,\chi}(g)(\alpha, a^2+\Delta) = \chi(-\delta y(g-1))(a\Delta, a^2+\Delta)\gamma(a^2+\Delta,\psi)$$

is independent of α. In particular, we may assume $\alpha = 1$. Taking $\beta = 0$ in the last lemma, we see that $\xi_{1,\chi}$ is a character of Γ_1 (since $\omega_{1,\chi}$ is a true representation).

Since Γ_1 is a pro-2 group and χ on $\mathcal{O}_E^\times$ has order five, $\chi(-\delta y(g-1)) = (-1)^{\mathrm{ord}_2(y)+\mathrm{ord}_2(g-1)}$.

Recall that (a, b) with $b \in \mathcal{O}^\times$ only depends on b mod 2^3, and when also $a \in \mathcal{O}^\times$, it only depends on b mod 2^2.

For $g \in \Gamma_3$, one has $1/a \in 4\mathcal{O}$, so $\mathrm{ord}_2(y) = 1 - \mathrm{ord}_2(a) = \mathrm{ord}_2(g-1)$, and $a^2 + \Delta = a^2(1 + \Delta/a^2)$ is a square, hence $\varphi(g) = 1$.

For $g \in \Gamma_3' - \Gamma_3$, one has $a \in 4\mathcal{O}$, so $\mathrm{ord}_2(y) = 1 + \mathrm{ord}_2(a)$ and $\mathrm{ord}_2(g-1) = 1$. Furthermore, $a^2 + \Delta = \Delta(1 + a^2/\Delta)$ is Δ times a square. Since $\gamma(\Delta,\psi) = 1$, and $(a,\Delta) = (-1)^{\mathrm{ord}_2(a)}$, we again get $\varphi(g) = 1$.

For $a = 1 + \sqrt{5}$, one has that $a^2 + \Delta$ is a square mod 4 but not mod 8 and $\mathrm{ord}_2(a) = 1$, so $(a\Delta, a^2 + \Delta) = -1$ and $\gamma(a^2 + \Delta, \psi) = \gamma(\Delta,\psi) = 1$. Since $\mathrm{ord}_2(y) = 2$, $\mathrm{ord}_2(g-1) = 1$, we have $\varphi(g_a) = 1$.

For $a = \frac{1-\sqrt{5}}{2}$, one has $a^2 + \Delta = -1$ and $a\Delta = -\sqrt{5}$, so $(a\Delta, a^2 + \Delta) = (-\sqrt{5}, -1) = -1$ and $\gamma(a^2 + \Delta, \psi) = \gamma(-1, \psi) = -1$. Since $\operatorname{ord}_2(y) = 1 = \operatorname{ord}_2(g - 1)$, we have $\varphi(g_a) = 1$.

Finally, for $a = 2$, one has $a^2 + \Delta = \left(\frac{-1+\sqrt{5}}{2}\right)^2$ and $\operatorname{ord}_2(y) = 2$, $\operatorname{ord}_2(g-1) = 1$, hence $\varphi(g_2) = -1$. □

Corollary 5.5. *If $\alpha \in \mathbb{Z}_2^\times$, then $f_{\frac{1}{2}}$ is an eigenfunction of Γ_1 with trivial character.*

Proof: By Lemma 5.3, it suffices to show that

$$\psi\Big(\frac{\Delta^2 \alpha y}{8x}\Big) = \psi\Big(\frac{\alpha a \Delta^2}{4(a^2 + \Delta)}\Big) = \begin{cases} 1 & \text{if } g \in \Gamma_3',\ a = 1 + \sqrt{5}, \text{ or } a = \frac{1-\sqrt{5}}{2}, \\ -1 & \text{if } a = 2. \end{cases}$$

This is trivially true for $g \in \Gamma_3'$ (since then the argument of ψ is integral). The other three cases are easily verified. □

Lemma 5.6. *For $\alpha \in \mathcal{O}^\times$ and $g = \zeta^j = x + y\delta$ with $5 \nmid j$, one has*

$$r(\iota_\alpha(g)) f_{\frac{1}{2}}(u) = \frac{1}{2}\psi\Big(\frac{\Delta \alpha x}{2y}(u^2 - \frac{u}{x} + \frac{1}{4})\Big) \operatorname{char}(\tfrac{1}{2}\mathcal{O})(u)\,.$$

Proof: This is a computation similar to that in the proof of Lemma 5.3, but using (5.2) instead of (5.1). Note that x and y are both in $\frac{1}{2}\mathcal{O}^\times$. □

We need to compute $\xi_{\alpha,\chi}(\zeta^j)$.

Lemma 5.7. *Let $\alpha \in \mathbb{Z}_2^\times$. Then*

$$\xi_{\alpha,\chi}(\zeta^j) = -\chi(\zeta^j - 1) \begin{cases} 1 & \text{if } j = \pm 1, \\ \pm\left(\frac{-1}{\alpha}\right) i & \text{if } j = \pm 2. \end{cases}$$

Furthermore, $\chi(\zeta^j - 1) = \zeta^{-(e+2)j}$, where $e = \operatorname{ord}_2(A)$.

Proof: Since $y \in F$ and $\operatorname{ord}_2(y) = -1$, we have $\chi(-\delta y) = -1$, which gives $\chi(-\delta y(\zeta^j - 1)) = -\chi(\zeta^j - 1)$. The factor involving the Weil index can be computed explicitly (use that $\gamma(a\psi)$ only depends on a mod 4 when a is a unit).

Finally, we have

$$\left(\frac{\zeta^j - 1}{2}\right)_5 \equiv (\zeta^j - 1)^3 \equiv 1 + \zeta^j + \zeta^{2j} + \zeta^{3j} \equiv \zeta^{-j} \bmod 2$$

and

$$\chi(\zeta^j - 1) = (-1)^e \left(\frac{\zeta^j - 1}{2}\right)_5^{e+2} = \left(\frac{\zeta^j - 1}{2}\right)_5^{e+2}$$

by Prop. 2.2, (4) and the fact that e is even (since $f_2 = 1$). □

Corollary 5.8. *Let $\alpha \in \mathbb{Z}_2^\times$ and denote $e = \mathrm{ord}_2(A)$. If we choose $\phi_2 = f_{\frac{1}{2}}$, then the constant C_2 in Prop. 3.1 has the value*

$$C_2 = \frac{1}{5}\Big(1 - \cos\big((e+2)\frac{2\pi}{5}\big) + \Big(\frac{-1}{\alpha}\Big)\sin\big(2(e+2)\frac{2\pi}{5}\big)\Big)\,,$$

and $I_2(x) = \phi_2(x)$.

Note that $C_2 \neq 0$ since $e \in \{0, 2, 4, 6\}$ when $f_2 = 1$.

Proof: By (3.1), we have to compute

$$C_2 = \int_G \langle \omega_{\alpha,\chi}(g) f_{\frac{1}{2}}, f_{\frac{1}{2}} \rangle \, dg\,.$$

By Cor. 5.5, $\omega_{\alpha,\chi}(g) f_{\frac{1}{2}} = f_{\frac{1}{2}}$ for $g \in \Gamma_1$, so $I_2 = \phi_2$, and by Lemma 5.2, (1), $G = \langle \zeta \rangle \times \Gamma_1$. Hence

$$\begin{aligned} C_2 &= \frac{1}{5}\sum_{j=0}^{4} \langle \omega_{\alpha,\chi}(\zeta^j) f_{\frac{1}{2}}, f_{\frac{1}{2}} \rangle \\ &= \frac{1}{5}\Big(1 + \tfrac{1}{2}\sum_{j=1}^{4} \xi_{\alpha,\chi}(\zeta^j)\psi\big(\frac{\Delta\alpha(x_j - 1)}{4y_j}\big)\Big) \end{aligned}$$

((by Lemma 5.6) where $\zeta^j = x_j + y_j\delta$)

$$\begin{aligned} &= \frac{1}{5}\Big(1 + \tfrac{1}{2}\xi_{\alpha,\chi}(\zeta) - \tfrac{1}{2}\xi_{\alpha,\chi}(\zeta^2) - \tfrac{1}{2}\xi_{\alpha,\chi}(\zeta^3) + \tfrac{1}{2}\xi_{\alpha,\chi}(\zeta^4)\Big) \\ &= \frac{1}{5}\Big(1 - \tfrac{1}{2}(\zeta^{e+2} + \zeta^{-(e+2)}) + \Big(\frac{-1}{\alpha}\Big)\tfrac{1}{2i}(\zeta^{2(e+2)} - \zeta^{-2(e+2)})\Big) \end{aligned}$$

(by Lemma 5.7)

$$= \frac{1}{5}\Big(1 - \cos\big((e+2)\frac{2\pi}{5}\big) + \Big(\frac{-1}{\alpha}\Big)\sin\big(2(e+2)\frac{2\pi}{5}\big)\Big)\,.$$

□

6. The Birch and Swinnerton-Dyer Conjecture

Since J_A has complex multiplication by $\mathbb{Z}[\zeta_5]$ defined over $E = \mathbb{Q}(\zeta_5)$, there is a Hecke character η_A of E such that

$$L(s, J_A) = L(s, \eta_A)\,,$$

compare [Sh1, Thm. 20.9]. See also [St2], where it is shown that η_A has conductor

$$\nu_A = \lambda^{f_\lambda} 2^{f_2} \prod_{2 \neq p \mid A} p$$

if A is not divisible by 5 or the tenth power of a prime number.

Recently, the Gross-Zagier formula [GZ] and the work of Kolyvagin and Logachev [Ko, KL] have been extended to totally real number fields by S.-W. Zhang [Zh1, Zh2] and Y. Tian [Ti] respectively under some technical conditions. If these conditions can be removed to cover more general cases, our result will imply finiteness of $J_1(\mathbb{Q})$ and of $Ш(\mathbb{Q}, J_1)$. In lack of those powerful results, our result still verifies the rank part of the Birch and Swinnerton-Dyer conjecture for J_1 since the finiteness of the Mordell-Weil group of J_1 can be proved by a 2-descent argument as in [St3]. In fact, $J_1(\mathbb{Q})$ has order 10 and is generated by the divisor class $[(0,1) - (-1,0)]$. The Birch and Swinnerton-Dyer conjecture would imply that $Ш(\mathbb{Q}, J_1)$ is actually trivial, see below. The 2-descent shows at least that its 2-part is trivial.

It is interesting to compare the L-series value $L(1, J_A)$ with the value predicted by the Birch and Swinnerton-Dyer conjecture. One version of it says that $L(1, J_A) \neq 0$ if and only if $J_A(\mathbb{Q})$ is finite, and then

$$L(1, J_A) = \frac{\Omega(J_A) \prod_p c_p(J_A) \# Ш(\mathbb{Q}, J_A)}{\# J_A(\mathbb{Q})^2}. \tag{6.1}$$

Here $\Omega(J_A)$ is the 'real period' of J_A over $\mathbb{Q}$ and $c_p(J_A)$ are the Tamagawa numbers. It is known by a general result of Blasius [Bl] that

$$\frac{L(1, J_A)}{\Omega(J_A)}$$

is rational. Since C_A has a rational point, $\# Ш(\mathbb{Q}, J_A)$ should be a square [PS]. So it is very interesting to see whether its predicted or "analytic" value for $A = 1$

$$\# Ш(\mathbb{Q}, J_1)_{\mathrm{an}} = \frac{L(1, J_1) \# J_1(\mathbb{Q})^2}{\Omega(J_1) \prod_p c_p(J_1)} = \frac{C\, |I|^2 \# J_1(\mathbb{Q})^2}{\Omega(J_1) \prod_p c_p(J_1)}$$

is indeed an integer square (C and I are given in Theorem 4.1). This involves relating the value of a theta function at a CM point to the square root of the period of J_1 in a precise way. G. Shimura proved in [Sh2] that the ratio is an algebraic number, which is not enough for our purpose. Pinning down which number field the ratio lies in is an important and non-trivial problem, since the period is in general defined only up to a scalar and thus the square root is defined only up to square roots of scalars.

We remark that all the invariants in (6.1) are computable except the order of the Shafarevich-Tate group, see [F+]. If one carries this through for the curves C_A, with A an integer not divisible by 5, one obtains the following.

Define

$$\omega(p, A) = p^{4\lfloor v_p(A)/10 \rfloor} \cdot \begin{cases} 1 & \text{if } v_p(A) \equiv 0, 1, 2, 3 \bmod 10; \\ p & \text{if } v_p(A) \equiv 4, 5, 6 \bmod 10; \\ p^2 & \text{if } v_p(A) \equiv 7, 8, 9 \bmod 10; \end{cases}$$

and

$$c(p, A) = \begin{cases} 5 & \text{if } 10 \nmid v_p(A) \text{ and } A \in (\mathbb{Q}_p^\times)^2; \\ 2 & \text{if } v_p(A) \equiv 5 \bmod 10 \text{ and } p \equiv \pm 2 \bmod 5; \\ 4 & \text{if } v_p(A) \equiv 5 \bmod 10 \text{ and } p \equiv -1 \bmod 5; \\ 16 & \text{if } v_p(A) \equiv 5 \bmod 10,\ A \in (\mathbb{Q}_p^\times)^5 \text{ and } p \equiv 1 \bmod 5; \\ 1 & \text{else.} \end{cases}$$

Then let

$$\omega_p(A) = \begin{cases} \omega(p, A) & \text{if } p \text{ is odd;} \\ \omega(2, A) & \text{if } p = 2 \text{ and } v_2(A) \text{ is odd;} \\ \omega(2, 2A) & \text{if } p = 2,\ v_2(A) \text{ is even and } 2^{-v_2(A)}A \equiv 3 \bmod 4; \\ \omega(2, 4A) & \text{if } p = 2,\ v_2(A) \text{ is even and } 2^{-v_2(A)}A \equiv 1 \bmod 4; \end{cases}$$

and similarly

$$c_p(A) = \begin{cases} c(p, A) & \text{if } p \neq 2, 5; \\ 1 & \text{if } p = 5 \text{ and } 5 \nmid q(A); \\ 2 & \text{if } p = 5 \text{ and } 5 \mid q(A); \\ c(2, A) & \text{if } p = 2 \text{ and } v_2(A) \text{ is odd;} \\ c(2, 2A) & \text{if } p = 2,\ v_2(A) \text{ is even and } 2^{-v_2(A)}A \equiv 3 \bmod 4; \\ c(2, 4A) & \text{if } p = 2,\ v_2(A) \text{ is even and } 2^{-v_2(A)}A \equiv 1 \bmod 4. \end{cases}$$

Also let

$$\Omega_0(A) = \frac{\pi \Gamma(\frac{1}{10}) \Gamma(\frac{3}{10})}{5 \Gamma(\frac{3}{5}) \Gamma(\frac{4}{5})} \cdot \begin{cases} A^{-2/5} & \text{if } A > 0; \\ (-A)^{-2/5}/\sqrt{5} & \text{if } A < 0. \end{cases}$$

Then we have

$$c_p(J_A) = c_p(A) \quad \text{and} \quad \Omega(J_A) = \Omega_0(A) \prod_p \omega_p(A)\,.$$

Furthermore,

$$\# J_A(\mathbb{Q})_{\mathrm{tors}} = \begin{cases} 10 & \text{if } A \in (\mathbb{Q}^\times)^{10}; \\ 5 & \text{if } A \in (\mathbb{Q}^\times)^2 \setminus (\mathbb{Q}^\times)^{10}; \\ 2 & \text{if } A \in (\mathbb{Q}^\times)^5 \setminus (\mathbb{Q}^\times)^{10}; \\ 1 & \text{else.} \end{cases}$$

For example, if $A = 1$, the Birch and Swinnerton-Dyer conjecture claims that

$$L(1, J_1) = \frac{\pi\Gamma(\frac{1}{10})\Gamma(\frac{3}{10})}{50\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})} \#\text{Ш}(\mathbb{Q}, J_1)$$

or equivalently

$$C|I|^2 = \frac{\pi\Gamma(\frac{1}{10})\Gamma(\frac{3}{10})}{50\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})} \#\text{Ш}(\mathbb{Q}, J_1).$$

Here C and I are given in Theorem 4.1. By numerical calculations, this would imply that $\text{Ш}(\mathbb{Q}, J_1)$ is trivial—the value for $\#\text{Ш}(\mathbb{Q}, J_1)$ comes out as 1.0 to many decimal digits. So the Birch and Swinnerton-Dyer conjecture in this case implies a relation between theta values and Gamma values.

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