GEOMETRY AND ARITHMETIC OF PRIMARY BURNIAT SURFACES

INGRID BAUER AND MICHAEL STOLL

ABSTRACT. We study the geometry and arithmetic of so-called *primary Burniat surfaces*, a family of surfaces of general type arising as smooth bidouble covers of a del Pezzo surface of degree 6 and at the same time as étale quotients of certain hypersurfaces in a product of three elliptic curves. We give a new explicit description of their moduli space and determine their possible automorphism groups. We also give an explicit description of the set of curves of geometric genus 1 on each primary Burniat surface. We then describe how one can try to obtain a description of the set of rational points on a given primary Burniat surface S defined over $\mathbb{Q}$. This involves an explicit description of the relevant twists of the étale covering of S coming from the second construction mentioned above and methods for finding the set of rational points on a given twist.

1. INTRODUCTION

Whereas it is well-known since Faltings' proof [Fal83] of the Mordell Conjecture that a curve of general type (and so of genus ≥ 2) over $\mathbb{Q}$ can have only finitely many rational points, not much is known in general for rational points on *surfaces* of general type. The Bombieri-Lang Conjecture predicts that the set $S(\mathbb{Q})$ of rational points on a surface S of general type is not Zariski dense; this is analogous to Faltings' theorem. This conjecture is open in general, but when S has a finite étale covering X that is a subvariety of an abelian variety, then its truth follows from a generalization of Faltings' result and descent theory, which we recall in Section 2. We can then ask whether it is possible to obtain an explicit description of $S(\mathbb{Q})$. The purpose of this paper is to initiate an investigation of this question by considering a special class of surfaces as a case study. The class of surfaces we look at are the so-called *primary Burniat surfaces*.

Primary Burniat surfaces are certain surfaces of general type first constructed by Burniat [Bur66] in the 1960s as smooth bidouble covers of a del Pezzo surface of degree 6. Their moduli space M forms a 4-dimensional irreducible connected component of the moduli space of surfaces of general type as shown by Mendes Lopes and Pardini [MLP01] using an argument related to the degree of the bicanonical map. This result was reproved by Bauer and Catanese in [BC11], where the authors also show the rationality of M. Their proof makes use of the fact (also proved in the same paper) that Burniat's surfaces can also be obtained as étale quotients of certain hypersurfaces in products of three elliptic curves, a construction first considered by Inoue [Ino94]. It is this description that allows us to show that Lang's Conjectures on rational points on varieties of general type hold for primary Burniat surfaces. In fact, we recall a more general statement for projective varieties admitting an étale covering that embeds in a product of abelian varieties and curves of genus at least two. This applies in particular for certain so-called Inoue type varieties; see Section 2.

After recalling the two different constructions of primary Burniat surfaces in Section 3, we produce an explicit description of their moduli space as a subset of $\mathbb{A}^6$ in Section 4. We then use this description to classify the possible automorphism groups of primary Burniat surfaces. Generically, this automorphism group is $(\mathbb{Z}/2\mathbb{Z})^2$, but its order can get as large as 96; see Section 5.

By the general theorem given in Section 2, we know that the set of rational points on a primary Burniat surface S consists of the points lying on the finitely many curves of geometric genus ≤ 1 on S, together with finitely many 'sporadic' points. To describe this set $S(\mathbb{Q})$ explicitly, we therefore need to determine the set of low-genus curves on S. This is done in Section 6. The result is that S does not contain rational curves, but does always contain six smooth curves of genus 1, and may contain up to six further (singular) curves of geometric genus 1.

In the last part of the paper, we discuss how the set $S(\mathbb{Q})$ can be determined explicitly for a primary Burniat surface S defined over $\mathbb{Q}$. We restrict to the case that the three genus 1 curves whose product contains the étale cover X of S are individually defined over $\mathbb{Q}$. We can then write down equations for X of a somewhat more general (but similar) form than those used in Section 4 where we were working over $\mathbb{C}$. We explain how to determine the finitely many twists of the covering $X \to S$ that may give rise to rational points on S, see Section 7, and we discuss what practical methods are available for the determination of the sets of rational points on these twists, see Section 8. We conclude with a number of examples.

2. Rational points on varieties of general type

We start giving a short account on some conjectures, which were formulated by S. Lang[1] [Lan86] after discussions with an input from several other mathematicians.

Conjecture 2.1 (Weak Lang Conjecture). *If X is a smooth projective variety of general type defined over a number field K, then the set $X(K)$ of K-rational points on X is not Zariski-dense in X.*

For X as in this conjecture, one defines the *exceptional set N* of X as the union of all the images of non-constant morphisms $f \colon A \to X$ from an abelian variety A to X, defined over $\bar{K}$.

Conjecture 2.2 (Strong Lang Conjecture). *Let X be a smooth projective variety of general type defined over a number field K. The exceptional set N of X is a proper closed subvariety of X, and for each finite field extension $K \subset L$, the set $X(L) \setminus N(L)$ is finite.*

In general these conjectures are wide open. They hold for curves (by Faltings' proof of the Mordell Conjecture), and more generally for subvarieties of abelian varieties, see [Fal94]. However, even in these cases it is far from clear that a suitable explicit description of $X(K)$ can actually be determined. In the case of curves, there has been considerable progress in recent years (see for example [Sto07]), indicating that an effective (algorithmic) solution might be possible.

It is therefore natural to consider the case of surfaces next, where so far only very little is known.

Let S be a smooth projective surface of general type over a number field K. In this case, Conjectures 2.1 and 2.2 state that there are only finitely many curves of geometric genus 0 or 1 on S (possibly defined over larger fields) and that the set of K-rational points on S outside these curves is finite. We call the curves of geometric genus at most 1 on S the *low-genus curves* on S, and we call the K-rational points outside the low-genus curves *sporadic K-rational points* on S.

An explicit description of $S(K)$ would then consist of the finite list of the low-genus curves on S, together with the finite list of sporadic K-rational points. Given a class of surfaces of general type, it is therefore natural to proceed in the following way:

(1) prove the finiteness of the set of low-genus curves on surfaces in the class;
(2) develop methods for determining this set explicitly;

---

[1] Sometimes these conjectures are referred to as 'Bombieri-Lang Conjectures', but Enrico Bombieri insists that he only ever made these conjectures for surfaces.

(3) prove the finiteness of the set of sporadic K-rational points on surfaces in the class;
(4) develop methods for determining the set of sporadic points explicitly.

Regarding point (1), the statement was proved by Bogomolov in the 1970s [Bog77] under the condition that the following inequality is satisfied:

$$c_1^2(S) = K_S^2 > c_2(S), \quad \text{or equivalently}, \quad K_S^2 > 6\chi(S).$$

Under the same assumption, Miyaoka [Miy08] gave an upper bound for the canonical degree $K_S \cdot C$ of a curve C of geometric genus ≤ 1 on S. A result by S. Lu [Lu10] gives a similar conclusion when the irregularity $q(S)$ is at least 2. For another approach that applies for example to diagonal varieties in arbitrary dimension, see [BM86, Vol85].

Results of Faltings [Fal94] and Kawamata [Kaw80, Theorem 4] imply that Conjecture 2.2 holds for subvarieties (of general type) of abelian varieties. Combining this with a generalization (see for example [Ser97, §4.2]) of a result due to Chevalley and Weil [CW32], this gives the following (see also [HS00, Prop. F.5.2.5 (ii)] for the 'weak' form).

Theorem 2.3. *Let X be a smooth projective variety of general type over a number field K which admits a finite étale covering $\pi \colon \hat{X} \to X$ such that $\hat{X}$ is contained in a product Z of abelian varieties and curves of higher genus (i.e., genus ≥ 2).*

(1) *$X(K)$ is not Zariski dense in X.*
(2) *If $\hat{X}$ does not contain any translate of a positive dimensional abelian subvariety of the abelian part of Z, then $X(K)$ is finite.*
(3) *X satisfies Conjecture 2.2.*

Remark 2.4. For the purpose of actually determining $X(K)$, we assume in addition that the covering π is geometrically Galois, so that $\hat{X}$ is an X-torsor under a finite K-group scheme G that we also assume to act on Z. (G is allowed to have fixed points on Z, but not on $\hat{X}$.) This allows us to use the machinery of descent (see for example [Sko01]). There is then a finite collection of twists $\pi_\xi \colon \hat{X}_\xi \to X$ of the torsor $\hat{X} \to X$ such that $X(K) = \bigcup_\xi \pi_\xi(\hat{X}_\xi(K))$. This collection of twists can be taken to be those that are unramified outside the set of places of K consisting of the archimedean places, the places dividing the order of G and the places of bad reduction for X or π.

Since G also acts on Z by assumption, we have $\hat{X}_\xi \subset Z_\xi$ for the corresponding twists of Z. The general form of such a twist is a product of a principal homogenous space for some abelian variety with some curves of higher genus or Weil restrictions of such curves over some finite extension of K (the latter can occur when the action of G on Z permutes some of the curve factors). Write $Z_\xi = Z_\xi^{(1)} \times Z_\xi^{(2)}$, where $Z_\xi^{(1)}$ is the 'abelian part' and $Z_\xi^{(2)}$ is the 'curve part'. By Faltings' theorem, $Z_\xi^{(2)}$

has only finitely many K-rational points. Fixing such a point P, the fiber $\hat{X}_{\xi,P}$ above P of $\hat{X}_\xi \to Z_\xi^{(2)}$ is a subvariety of the principal homogeneous space $Z_\xi^{(1)}$. So either $\hat{X}_{\xi,P}(K)$ is empty, or else we can consider $Z_\xi^{(1)}$ as an abelian variety, so that Faltings' theorem applies to $\hat{X}_{\xi,P}$. This shows that $\hat{X}_{\xi,P}(K)$ is contained in a finite union of translates of abelian subvarieties of $Z_\xi^{(1)}$ that are contained in $\hat{X}_{\xi,P}$. Since there are only finitely many P, the corresponding statement holds for $\hat{X}_\xi$ as well.

The advantage of this variant is that it allows us to work with K-rational points on $\hat{X}$ and its twists, rather than with K'-rational points on $\hat{X}$ for a (usually rather large) extension K' of K. So if we have a way of determining the translates of abelian subvarieties of positive dimension contained in $\hat{X}$ and also of finding the finitely many K-rational points outside these translates for each twist of $\hat{X}$, we can determine $X(K)$. We will pursue this approach when X is a primary Burniat surface in Sections 7, 8 and 9 below.

Theorem 2.3 applies to a generalization of Inoue's construction of Burniat's surfaces, the so-called *classical Inoue type varieties*, which were introduced in [BC12], and which proved to be rather useful in the study of deformations in the large of surfaces of general type. In fact, it turned out that for Inoue type varieties one can show under certain additional conditions that any smooth variety homotopically equivalent to an Inoue type variety is in fact an Inoue type variety. This often allows to determine and describe the connected components of certain moduli spaces of surfaces of general type, since it implies that a family yields a closed subset in the moduli space. For details we refer to the original article [BC12]. A classical Inoue type variety is of general type, and if it is defined over a number field K, then it satisfies the assumptions of Theorem 2.3 and also Remark 2.4. Taking $K = \mathbb{Q}$ (to keep the computations feasible), we obtain a large class of examples of varieties of general type for which one can try to use the approach of Remark 2.4 to determine the set of rational points explicitly.

3. Primary Burniat Surfaces

The so-called *Burniat surfaces* are several families of surfaces of general type with $p_g = 0$, $K^2 = 6, 5, 4, 3, 2$, first constructed by P. Burniat in [Bur66] as singular *bidouble covers* (i.e., $(\mathbb{Z}/2\mathbb{Z})^2$ Galois covers) of the projective plane $\mathbb{P}^2$ branched on a certain configuration of nine lines. Later, M. Inoue [Ino94] gave another construction of surfaces 'closely related to Burniat's surfaces' with a different technique as $(\mathbb{Z}/2\mathbb{Z})^3$-quotients of an invariant hypersurface of multi-degree $(2, 2, 2)$ in a product of three elliptic curves. In [BC11] it is shown that these two constructions give exactly the same surfaces.

In fact, concerning statements about rational points on Burniat surfaces, Inoue's construction is much more useful than the original construction of Burniat, as will be clear soon.

We briefly recall Burniat's original construction of Burniat surfaces with $K^2 = 6$, which were called *primary* Burniat surfaces in [BC11].

Consider three non-collinear points $p_1, p_2, p_3 \in \mathbb{P}^2$, which we take to be the co-ordinate points, and denote by $Y := \hat{\mathbb{P}}^2(p_1, p_2, p_3)$ the blow-up of $\mathbb{P}^2$ in p_1, p_2, p_3. Then Y is a del Pezzo surface of degree 6 and it is the closure of the graph of the rational map

$$\epsilon \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

such that

$$\epsilon(y_1 : y_2 : y_3) = \big((y_2 : y_3), (y_3 : y_1), (y_1 : y_2)\big).$$

We denote by e_i the exceptional curve lying over p_i and by $D_{i,1}$ the unique effective divisor in $|l - e_i - e_{i+1}|$, i.e., the proper transform of the line $y_{i-1} = 0$, side of the triangle joining the points p_i, p_{i+1}. Here l denotes the total transform of a general line in $\mathbb{P}^2$.

Consider on Y the divisors

$$D_i = D_{i,1} + D_{i,2} + D_{i,3} + e_{i+2} \in |3l - 3e_i - e_{i+1} + e_{i+2}|,$$

where $D_{i,j} \in |l - e_i|$, for $j = 2, 3$, $D_{i,j} \neq D_{i,1}$, is the proper transform of another line through p_i and $D_{i,1} \in |l - e_i - e_{i+1}|$ is as above. Assume also that all the corresponding lines in $\mathbb{P}^2$ are distinct, so that $D := \sum_i D_i$ is a reduced divisor. In this description, the subscripts are understood as residue classes modulo 3.

If we define the divisor $L_i := 3l - 2e_{i-1} - e_{i+1}$, then

$$D_{i-1} + D_{i+1} \equiv 6l - 4e_{i-1} - 2e_{i+1} = 2L_i,$$

and we can consider the associated bidouble cover $S \to Y$ branched on D.

From the birational point of view, the function field of S is obtained from the function field of $\mathbb{P}^2$ by adjoining

$$\sqrt{\frac{\Delta_1}{\Delta_3}}, \; \sqrt{\frac{\Delta_2}{\Delta_3}},$$

where $\Delta_i \in \mathbb{C}[x_0, x_1, x_2]$ is the cubic polynomial whose zero set has strict transform D_i.

A biregular model is given by the following: let $D_i = \mathrm{div}(\delta_i)$ and let u_i be a fibre coordinate of the geometric line bundle $\mathbb{L}_i$ whose sheaf of holomorphic sections is

$\mathcal{O}_Y(L_i)$. Then $S \subset \mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_3$ is given by the equations:

$$u_1 u_2 = \delta_1 u_3 \ , \ \ u_1^2 = \delta_3 \delta_1,$$
$$u_2 u_3 = \delta_2 u_3 \ , \ \ u_2^2 = \delta_1 \delta_2,$$
$$u_3 u_1 = \delta_3 u_2 \ , \ \ u_3^2 = \delta_2 \delta_3 \ .$$

This shows clearly that $S \to Y$ is a Galois cover with group $(\mathbb{Z}/2\mathbb{Z})^2$ and we see the intermediate double covers as the double covers of Y branched on $D_i + D_j$. Compare Figure 1.
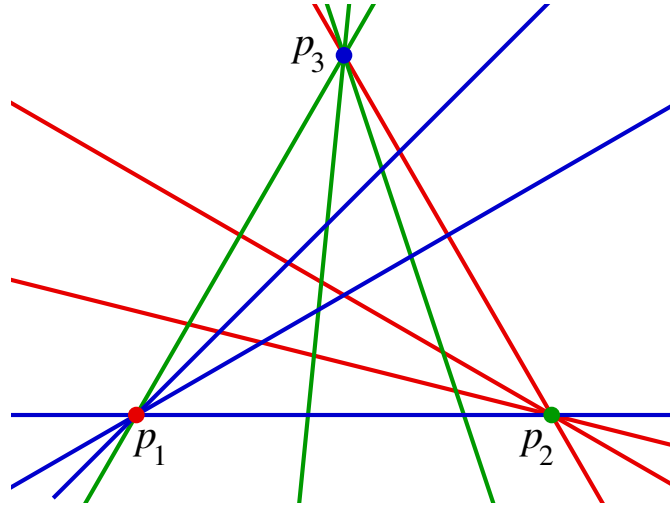

FIGURE 1. The configuration of the branch locus in Burniat's construction. D_1 is blue, D_2 is red and D_3 is green.

Observe that S is nonsingular exactly when the divisor D does not have points of multiplicity 3 (there cannot be points of higher multiplicities).

Definition 3.1. A *primary Burniat surface* is a surface S as constructed above that is non-singular.

Remark 3.2. A primary Burniat surface S is a minimal surface with K_S ample. It has invariants $K_S^2 = 6$ and $p_g(S) = q(S) = 0$.

We now describe Inoue's construction in detail, since we shall make extensive use of it. We pick three elliptic curves E_1, E_2, E_3, together with points $T_j \in E_j$ of order 2. On each E_j, we then have two commuting involutions, namely the translation by T_j and the negation map. We now consider the group $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ of

automorphisms of the product $E_1 \times E_2 \times E_3$ with

$$\gamma_1 \colon (P_1, P_2, P_3) \longmapsto (P_1, P_2 + T_2, -P_3 + T_3)$$
$$\gamma_2 \colon (P_1, P_2, P_3) \longmapsto (-P_1 + T_1, P_2, P_3 + T_3)$$
$$\gamma_3 \colon (P_1, P_2, P_3) \longmapsto (P_1 + T_1, -P_2 + T_2, P_3).$$

It can be checked that none of the γ_j and none of the $\gamma_i\gamma_j$ (with $i \neq j$) have fixed points on $E_1 \times E_2 \times E_3$. However, the automorphism $\gamma = \gamma_1\gamma_2\gamma_3$ is given by $(P_1, P_2, P_3) \mapsto (-P_1, -P_2, -P_3)$, so it has 4^3 fixed points, namely the points (P_1, P_2, P_3) such that P_j is a 2-torsion point on E_j for each j.

Now let x_j be a function of degree 2 on E_j that is invariant under the negation map and changes sign under translation by T_j. Then the product $x_1x_2x_3$ is invariant under Γ, and so the surface $X \subset E_1 \times E_2 \times E_3$ defined by $x_1x_2x_3 = c$ is also Γ-invariant.

Remark 3.3. It can be checked that X is smooth unless it contains a fixed point of γ or $c = 0$ [Ino94]. So as long as we avoid a finite set of values for c, the surface X will be smooth and Γ acts on it without fixed points. We let $S = X/\Gamma$ be the quotient. Then $\pi\colon X \to S$ is an unramified covering that is Galois with Galois group Γ. Both X and S are surfaces of general type.

In [BC11, Theorem 2.3] the following is shown:

Theorem 3.4. *Primary Burniat surfaces are exactly the surfaces $S = X/\Gamma$ as above such that Γ acts freely on X.*

Remark 3.5. This implies that primary Burniat surfaces are classical Inoue type varieties.

4. An explicit moduli space for primary Burniat surfaces

In [BC11, Theorem 4.1, Theorem 4.2] the following result is shown:

Theorem 4.1.

(1) *Let S be a smooth complex projective surface that is homotopically equivalent to a primary Burniat surface. Then S is a primary Burniat surface.*
(2) *The subset M of the Gieseker moduli space corresponding to primary Burniat surfaces is an irreducible connected component, normal and rational of dimension 4.*

In particular, this shows that if $[S] \in M$, then S is in fact a primary Burniat surface, i.e., it can be obtained via Burniat's and Inoue's constructions.

Still, in [BC11] no explicit model of M is given. The goal of this section is to provide such an explicit moduli space based on Inoue's construction. As a by-product, we obtain another proof of its rationality. We will also use our approach

to classify the possible automorphism groups of primary Burniat surfaces, see Section 5 below. We work over $\mathbb{C}$.

We first consider a curve E of genus 1 given as a double cover $\pi_E \colon E \to \mathbb{P}^1$ ramified in four points, together with a point $T \in J$ of order 2, where J is the Jacobian elliptic curve of E (so that E is a principal homogeneous space for J; more precisely, E is a 2-covering of J via $P \mapsto P - \iota_E(P)$, where ι_E is the involution on E induced by π_E). There is then a coordinate x on $\mathbb{P}^1$, unique up to scaling and replacing x by $1/x$, such that $x(\pi_E(P + T)) = -x(\pi_E(P))$ for all $P \in E$. (To see the uniqueness, fix a point $Q \in E$ with $Q - \iota_E(Q) = T$. Then the divisor of $x \circ \pi_E$ must be $\pm\big((Q) + (\iota_E(Q)) - (Q + T') - (\iota_E(Q) + T')\big)$, where $T' \neq T$ is another point of order 2.) As a double cover of $\mathbb{P}^1$ via $x \circ \pi_E$, E is then given by an equation of the form $y^2 = x^4 + ax^2 + b$. We fix the scaling of x up to a fourth root of unity by requiring $b = 1$. This fixes a up to a sign. So the moduli space of pairs $(E \to \mathbb{P}^1, T)$ as above (i.e., $E \to \mathbb{P}^1$ is a double cover and T is a point of order 2 on the Jacobian of E) is $\mathbb{A}^1 \setminus \{4\}$, with the parameter for $y^2 = x^4 + ax^2 + 1$ being a^2. If we identify E with J by taking the origin to be one of the ramification points of x, then y changes sign under negation. In general, the fibers $\{P, P'\}$ of $\pi_E \colon J \cong E \to \mathbb{P}^1$ are characterized by $P + P' = Q$ for some point $Q \in J$; then the automorphism ι_E of E corresponding to changing the sign of y is given by $P \mapsto Q - P$.

Consider the subvariety $\mathcal{E}$ given by $y^2 = x^4 + ax^2z^2 + z^4$ in $\mathbb{P}(1, 2, 1) \times (\mathbb{A}^1 \setminus \{\pm 2\})$, where x, y, z are the coordinates of $\mathbb{P}(1, 2, 1)$ and a is the coordinate on $\mathbb{A}^1$ (then x/z is the function denoted x in the previous paragraph). Then by the above, the group of automorphisms of $\mathcal{E}$ that respect the morphism to $\mathbb{P}^1$ given by $(x : z)$ and are compatible with the fibration over $\mathbb{A}^1 \setminus \{\pm 2\}$ is generated by

$$
\begin{aligned}
\big((x : y : z), a\big) &\longmapsto \big((ix : y : z), -a\big), \\
\big((x : y : z), a\big) &\longmapsto \big((x : -y : z), a\big) \quad \text{and} \\
\big((x : y : z), a\big) &\longmapsto \big((z : y : x), a\big).
\end{aligned}
$$

The fibers $(E_a \to \mathbb{P}^1, T_a)$ and $(E_b \to \mathbb{P}^1, T_b)$ over a and b are isomorphic (in the sense that there is an isomorphism $E_a \to E_b$ that induces an automorphism of $\mathbb{P}^1$ and identifies T_a with T_b) if and only if $b = \pm a$, and $(E_a \to \mathbb{P}^1, T_a)$ has extra automorphisms if and only if $a = 0$.

We now use a similar approach to classify surfaces $X \subset E_1 \times E_2 \times E_3$ as in Inoue's construction of primary Burniat surfaces. By the above, we can define X as a

subvariety of $E_1 \times E_2 \times E_3 \subset \mathbb{P}(1, 2, 1)^3$ by equations

(4.1)
$$
\begin{cases}
y_1^2 = x_1^4 + a_1 x_1^2 z_1^2 + z_1^4 \\
y_2^2 = x_2^4 + a_2 x_2^2 z_2^2 + z_2^4 \\
y_3^2 = x_3^4 + a_3 x_3^2 z_3^2 + z_3^4 \\
x_1 x_2 x_3 = c z_1 z_2 z_3
\end{cases}
$$

with suitable parameters $a_1, a_2, a_3 \neq \pm 2$ and $c \neq 0$. Here x_j / z_j is the coordinate on E_j that changes sign under addition of T_j (denoted x above). We have to determine for which choices of the parameters (a_1, a_2, a_3, c) we obtain isomorphic surfaces X/Γ.

Let $A = \left(\mathbb{A}^1 \setminus \{\pm 2\} \right)^3 \times \left(\mathbb{A}^1 \setminus \{0\} \right)$ and consider $\mathcal{X} \subset \mathbb{P}(1, 2, 1)^3 \times A$ as given by equations (4.1), where x_j, y_j, z_j are the coordinates on the three factors $\mathbb{P}(1, 2, 1)$ (for $j = 1, 2, 3$) and a_1, a_2, a_3, c are the coordinates on A. Then the group of automorphisms of $\mathbb{P}(1, 2, 1)^3 \times A$ that fix $\mathcal{X}$ and are compatible with the fibration over A is generated by the following automorphisms, which are specified by the image of the point

$$
\left((x_1 : y_1 : z_1), (x_2 : y_2 : z_2), (x_3 : y_3 : z_3), (a_1, a_2, a_3, c) \right).
$$

$$
\begin{array}{llll}
\rho_1 : & \left((ix_1 : y_1 : z_1), \right. & (x_2 : y_2 : z_2), & (x_3 : y_3 : z_3), & \left. (-a_1, a_2, a_3, ic) \right) \\
\rho_2 : & \left((x_1 : y_1 : z_1), \right. & (ix_2 : y_2 : z_2), & (x_3 : y_3 : z_3), & \left. (a_1, -a_2, a_3, ic) \right) \\
\rho_3 : & \left((x_1 : y_1 : z_1), \right. & (x_2 : y_2 : z_2), & (ix_3 : y_3 : z_3), & \left. (a_1, a_2, -a_3, ic) \right) \\
\gamma_1' : & \left((x_1 : -y_1 : z_1), \right. & (x_2 : y_2 : z_2), & (x_3 : y_3 : z_3), & \left. (a_1, a_2, a_3, c) \right) \\
\gamma_2' : & \left((x_1 : y_1 : z_1), \right. & (x_2 : -y_2 : z_2), & (x_3 : y_3 : z_3), & \left. (a_1, a_2, a_3, c) \right) \\
\gamma_3' : & \left((x_1 : y_1 : z_1), \right. & (x_2 : y_2 : z_2), & (x_3 : -y_3 : z_3), & \left. (a_1, a_2, a_3, c) \right) \\
\tau : & \left((z_1 : y_1 : x_1), \right. & (z_2 : y_2 : x_2), & (z_3 : y_3 : x_3), & \left. (a_1, a_2, a_3, 1/c) \right) \\
\sigma : & \left((x_2 : y_2 : z_2), \right. & (x_3 : y_3 : z_3), & (x_1 : y_1 : z_1), & \left. (a_2, a_3, a_1, c) \right) \\
\tau' : & \left((x_2 : y_2 : z_2), \right. & (x_1 : y_1 : z_1), & (x_3 : y_3 : z_3), & \left. (a_2, a_1, a_3, c) \right)
\end{array}
$$

Note that $\Gamma = \langle \rho_1^2 \rho_2^2 \gamma_2', \rho_2^2 \rho_3^2 \gamma_3', \rho_3^2 \rho_1^2 \gamma_1' \rangle$ fixes $\mathcal{X}$ fiber-wise. Since we are interested in the quotients of the fibers by this action, we restrict to the subgroup consisting of elements normalizing Γ. This means that we exclude τ'.

So let $\tilde{\Gamma} = \langle \rho_1, \rho_2, \rho_3, \gamma_1', \gamma_2', \gamma_3', \tau, \sigma \rangle$. The first six elements generate an abelian normal subgroup isomorphic to $(\mathbb{Z}/4\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3$ on which τ acts by negation and σ acts by cyclically permuting the factors in both $(\mathbb{Z}/4\mathbb{Z})^3$ and $(\mathbb{Z}/2\mathbb{Z})^3$. In particular, $\tilde{\Gamma}$ has order $4^3 \cdot 2^3 \cdot 2 \cdot 3 = 2^{10} \cdot 3$. We note that the kernel of the natural homomorphism $\tilde{\Gamma} \to \operatorname{Aut}(A)$ is $\tilde{\Gamma}_0 := \langle \Gamma, \gamma_1', \gamma_2' \rangle$ (we also have $\gamma_3' = \gamma \gamma_1' \gamma_2' \in \tilde{\Gamma}_0$). So on the quotient S of every fiber X by Γ, we have an action of $\tilde{\Gamma}_0 / \Gamma$, and the quotient of S by this action is the del Pezzo surface Y in Burniat's construction.

The image $\tilde{\Gamma}_A$ of $\tilde{\Gamma}$ in $\mathrm{Aut}(A)$ is generated by

$$\bar\rho_1\colon (a_1, a_2, a_3, c) \longmapsto (-a_1, a_2, a_3, ic)$$
$$\bar\rho_2\colon (a_1, a_2, a_3, c) \longmapsto (a_1, -a_2, a_3, ic)$$
$$\bar\rho_3\colon (a_1, a_2, a_3, c) \longmapsto (a_1, a_2, -a_3, ic)$$
$$\bar\tau\colon (a_1, a_2, a_3, c) \longmapsto (a_1, a_2, a_3, 1/c)$$
$$\bar\sigma\colon (a_1, a_2, a_3, c) \longmapsto (a_2, a_3, a_1, c)$$

Note that $\zeta := \bar\rho_1^2 = \bar\rho_2^2 = \bar\rho_3^2$ just changes the sign of c. The order of $\tilde{\Gamma}_A$ is 96. There are twenty conjugacy classes. The table below gives a representative element, its order and the size of each class.

rep.	1	ζ	$\bar\rho_1$	$\bar\rho_1\bar\rho_2$	$\bar\rho_1\bar\rho_2^{-1}$	$\bar\rho_1\bar\rho_2\bar\rho_3$	$\bar\sigma$	$\zeta\bar\sigma$	$\bar\rho_1\bar\sigma$	$\bar\sigma^2$
order	1	2	4	2	2	4	3	6	12	3
size	1	1	6	3	3	2	4	4	8	4

rep.	$\zeta\bar\sigma^2$	$\bar\rho_1\bar\sigma^2$	$\bar\tau$	$\bar\rho_1\bar\tau$	$\bar\rho_1\bar\rho_2\bar\tau$	$\bar\rho_1\bar\rho_2\bar\rho_3\bar\tau$	$\bar\sigma\bar\tau$	$\bar\rho_1\bar\sigma\bar\tau$	$\bar\sigma^2\bar\tau$	$\bar\rho_1\bar\sigma^2\bar\tau$
order	6	12	2	2	2	2	6	6	6	6
size	4	8	2	6	6	2	8	8	8	8

Given a primary Burniat surface S, the $(\mathbb{Z}/2\mathbb{Z})^3$-cover $\pi\colon X \to S$ with $X \subset E_1 \times E_2 \times E_3$ is uniquely determined up to isomorphism as the unramified covering corresponding to the unique normal abelian subgroup of $\pi_1(S)$ such that the quotient is $(\mathbb{Z}/2\mathbb{Z})^3$. (See [BC11, Section 3] for an explicit description of $\pi_1(S)$, which implies this.) It follows that the (coarse) moduli space M of primary Burniat surfaces is $A'/\tilde{\Gamma}_A$, where A' is the open subset of A over which the fibers of $\mathcal{X}$ do not pass through a fixed point of γ. We determine the invariants for this group action. We have the normal subgroup $\langle \zeta, \bar\tau \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$ that fixes (a_1, a_2, a_3) and has orbit $\{c, -c, 1/c, -1/c\}$ on c. The invariants for this action are a_1, a_2, a_3 and $v' := c^2 + 1/c^2$. The action of $\bar\rho_j$ on these invariants is given by changing the signs of a_j and v', so the invariants of the action by the group generated by the $\bar\rho_j$ and $\bar\tau$ are a_1^2, a_2^2, a_3^2, $v := v'^2$ and $w := a_1 a_2 a_3 v'$. Finally, $\bar\sigma$ fixes v and w and permutes a_1^2, a_2^2, a_3^2 cyclically. So the invariants of the action of $\tilde{\Gamma}_A$ are the elementary symmetric polynomials in the a_j^2, namely

$$u_1 = a_1^2 + a_2^2 + a_3^2, \quad u_2 = a_1^2 a_2^2 + a_2^2 a_3^2 + a_3^2 a_1^2, \quad u_3 = a_1^2 a_2^2 a_3^2,$$
$$\text{together with} \quad d = (a_1^2 - a_2^2)(a_2^2 - a_3^2)(a_3^2 - a_1^2) \quad \text{and } v, w.$$

This exhibits the moduli space M as an open subset of a bidouble cover of $\mathbb{A}^4$ (with coordinates u_1, u_2, u_3, v) as follows.

Theorem 4.2. *The moduli space M of primary Burniat surfaces is the subset of $\mathbb{A}^6$ defined by the following equations and inequalities.*

$$-4u_1^3 u_3 + u_1^2 u_2^2 + 18 u_1 u_2 u_3 - 4u_2^3 - 27 u_3^2 = d^2$$

$$u_3 v = w^2$$

$$(v - u_1)^2 + u_2(v - 4) + u_3 + (u_1 + v - 8)w \neq 0$$

$$64 - 16u_1 + 4u_2 - u_3 \neq 0$$

Proof. This follows from the discussion above. Note that the expression on the left hand side of the first equation is the discriminant of $X^3 - u_1 X^2 + u_2 X - u_3$. The first inequality is equivalent to the non-vanishing of the discriminant of X as defined in Section 7 below and the second inequality encodes the non-singularity of the curves $y_j^2 = x_j^4 + a_j x_j^2 z_j^2 + z_j^4$ (which is equivalent to $a_j^2 \neq 4$). $\qquad\square$

Corollary 4.3. *M is rational.*

Proof. First note that M is birational to the product of $\mathbb{A}^1$ with the double cover of $\mathbb{A}^3$ given by the first equation (we can eliminate v using the second equation, and then w is a free variable). The first equation is equivalent to

$$4(3u_2 - u_1^2)^3 = (27u_3 + 2u_1^3 - 9u_1 u_2)^2 + 27d^2 \,.$$

By a coordinate change, we reduce to the product of $\mathbb{A}^2$ with the rational surface given by $x^3 + y^2 + z^2 = 0$. $\qquad\square$

5. Automorphism groups of primary Burniat surfaces

We now consider the group of automorphisms of a primary Burniat surface S. Since the covering $\pi\colon X \to S$ is unique, every automorphism of S lifts to an element of $\tilde{\Gamma}$ fixing X. We have already seen that $\mathrm{Aut}(S)$ always contains $\bar{\Gamma}_0 := \tilde{\Gamma}_0/\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^2$; since $\tilde{\Gamma}_0$ is normal in $\tilde{\Gamma}$, this is a normal subgroup of $\mathrm{Aut}(S)$.

Additional automorphisms of X correspond to elements of $\tilde{\Gamma}$ that fix the parameters (a_1, a_2, a_3, c). There are the following possibilities for such elements modulo $\tilde{\Gamma}_0$. We give their images in $\tilde{\Gamma}_A$, up to conjugation. Note that ζ, $\bar{\rho}_1$, $\bar{\rho}_1\bar{\rho}_2\bar{\rho}_3$, $\bar{\rho}_1\bar{\rho}_2$, $\zeta\bar{\sigma}$, $\bar{\rho}_1\bar{\sigma}$, $\zeta\bar{\sigma}^2$, $\bar{\rho}_1\bar{\sigma}^2$ map c to $\pm ic$ or $-c$ and cannot fix any $c \neq 0$. Also, $\bar{\rho}_1\bar{\rho}_2\bar{\rho}_3\bar{\tau}$ can only fix (a_1, a_2, a_3, c) when $a_1 = a_2 = a_3 = 0$ and $c^4 + 1 = 0$, which is excluded (it corresponds to a singular X).

(i) $\bar{\tau}$, corresponding to $c = 1/c$; this is $v = 4$ on M.
Then $\mathrm{Aut}(S)$ acquires an extra involution τ' that commutes with $\bar{\Gamma}_0$. We write M_1 for the corresponding subvariety of M.
(ii) $\bar{\rho}_1\bar{\rho}_2^{-1}$, corresponding to $a_1 = a_2 = 0$; this is $u_2 = u_3 = 0$ on M, which implies $d = w = 0$.

Then $\mathrm{Aut}(S)$ acquires an element ρ' of order 4 that commutes with $\bar{\Gamma}_0$ and whose square is in $\bar{\Gamma}_0$. We write M_2 for the corresponding subvariety of M.

(iii) $\bar{\rho}_1\bar{\tau}$, corresponding to $a_1 = 0$ and $c = i/c$; this is $u_3 = v = 0$ on M, which implies $w = 0$.

Then $\mathrm{Aut}(S)$ acquires an extra involution τ'' that commutes with $\bar{\Gamma}_0$. We write M_3 for the corresponding subvariety of M.

(iv) $\bar{\sigma}$, corresponding to $a_1 = a_2 = a_3$; this is $d = 0$, $u_1^2 = 3u_2$, $u_1 u_2 = 9u_3$ on M. Then $\mathrm{Aut}(S)$ acquires an extra element σ' of order 3 that acts non-trivially on $\bar{\Gamma}_0$. The subgroup of $\mathrm{Aut}(S)$ generated by $\bar{\Gamma}_0$ and the additional automorphism is isomorphic to the alternating group A_4. We write M_4 for the corresponding subvariety of M.

By considering the action on (a_1, a_2, a_3, c), it can be checked that elements of other conjugacy classes can only be present when we are in one of the four cases (i)–(iv) above. More precisely, of the 14 subgroups up to conjugacy that do not contain one of the 'forbidden' elements given above, four cannot occur, since the action implies that there is actually a larger group present. The ten remaining groups correspond to those given below.

Note that $M_1 \cap M_3 = M_3 \cap M_4 = \emptyset$. We obtain the following stratification of M. The leftmost column gives the dimension.
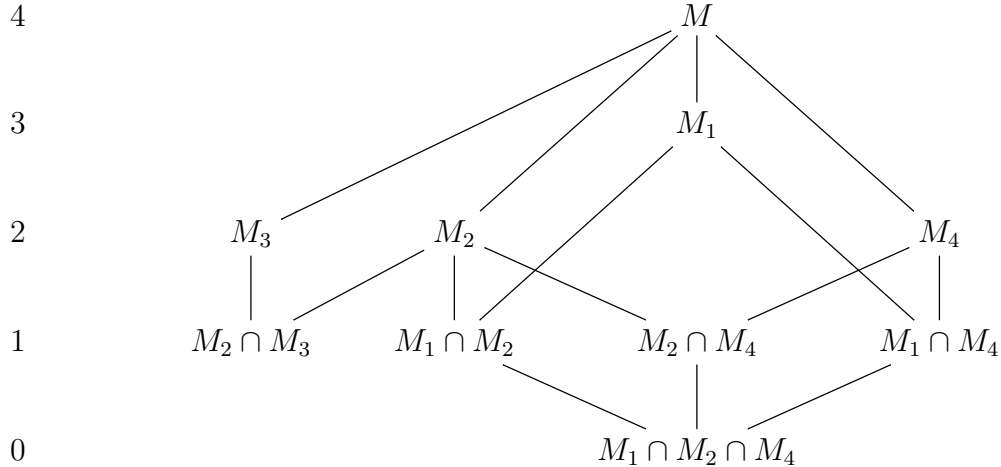
Note that $M_1 \cap M_2 \cap M_4$ is the single point $(u_1, u_2, u_3, d, v, w) = (0, 0, 0, 0, 4, 0)$, which is represented by $(a_1, a_2, a_3, c) = (0, 0, 0, 1)$.

The following table specifies $\mathrm{Aut}(S)$ and $\mathrm{Aut}(S)/\bar{\Gamma}'$ for the various strata. We write C_n for a cyclic group of order n and D_4 for the dihedral group of order 8. For a group G, $(G \wr C_3)/G$ denotes the wreath product $G^3 \rtimes C_3$ divided by the diagonal image of G in the normal subgroup G^3. Note that $A_4 \cong (C_2 \wr C_3)/C_2$. The semidirect product $(C_4 \wr C_3)/C_4 \rtimes C_2$ in the last line is such that the nontrivial element of C_2 (which is τ') acts on $(C_4 \wr C_3)/C_4$ by simultaneously inverting the

elements in the C_4 components. The table can be checked by observing that $\tau'^2 = \tau''^2 = 1$, $\rho'^2 \in \bar{\Gamma}_0$, $\rho'^4 = 1$, $\tau'\rho' = \rho'^{-1}\tau'$, $\tau''\rho' = \rho'^{-1}\tau''$, $\tau'\sigma' = \sigma'\tau'$, $(\rho'\sigma')^3 = 1$, and τ', ρ', τ'' commute with $\bar{\Gamma}_0$, whereas σ' acts non-trivially on it.

stratum	$\mathrm{Aut}(S)$	$\mathrm{Aut}(S)/\bar{\Gamma}'$
M	C_2^2	C_1
M_1	C_2^3	C_2
M_2	$C_2 \times C_4$	C_2
M_3	C_2^3	C_2
M_4	A_4	C_3
$M_1 \cap M_2$	$C_2 \times D_4$	C_2^2
$M_1 \cap M_4$	$C_2 \times A_4$	C_6
$M_2 \cap M_3$	$C_2 \times D_4$	C_2^2
$M_2 \cap M_4$	$(C_4 \wr C_3)/C_4$	A_4
$M_1 \cap M_2 \cap M_4$	$(C_4 \wr C_3)/C_4 \rtimes C_2$	$A_4 \times C_2$

6. Low genus curves on primary Burniat surfaces

We now study the set of curves of geometric genus 0 or 1 on a primary Burniat surface S. This is more easily done using Inoue's construction of S as an étale quotient of a hypersurface X in a product $E_1 \times E_2 \times E_3$ of three elliptic curves. Since X does not contain rational curves, the same is true for S. So it remains to find the curves of geometric genus 1 on S. Such curves are images of smooth curves of genus 1 contained in X under the quotient map $\pi\colon X \to S$, so we need to classify these.

Let $E \subset X$ be a curve of genus 1. The three projections of $E_1 \times E_2 \times E_3$ give us three morphisms $\psi_j\colon E \to X \to E_j$, which must be translations followed by isogenies or constant. There are three cases. Note that at least one of these morphisms must be non-constant.

6.1. **Two of the ψ_j are constant.** Say ψ_2 and ψ_3 are constant, with values $(\xi_2 : \eta_2 : \zeta_2)$ and $(\xi_3 : \eta_3 : \zeta_3)$. If $\xi_2\xi_3 \neq 0$, this would force $x_1 = c(\zeta_2\zeta_3/\xi_2\xi_3)z_1$. and ψ_1 would have to be constant as well. So $\xi_2 = 0$ (say). Since $c \neq 0$, we must then have $\zeta_3 = 0$. This gives two choices for the image of ψ_2 and two choices for the image of ψ_3; by interchanging the roles of ξ and ζ, we obtain four further choices. This gives rise to eight copies of E_1 in X that are pairwise disjoint. Under the action of Γ, they form two orbits of size four, so on S we obtain two copies of $E_1/\langle T_1 \rangle$. In the same way, we have two copies each of $E_2/\langle T_2 \rangle$ and of $E_3/\langle T_3 \rangle$. These six smooth curves of genus 1 are arranged in the form of a hexagon, with the two images of E_j corresponding to opposite sides, see Figure 2. In Burniat's original construction, these curves are obtained as preimages of the

three lines joining the three points that are blown up and the three exceptional divisors obtained from blowing up the points.

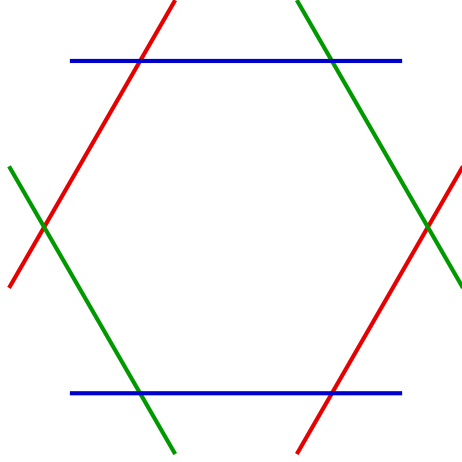**Definition 6.1.** We call these six curves the *curves at infinity* on S.

FIGURE 2. The six curves at infinity.

6.2. **Exactly one of the ψ_j is constant.** Say ψ_1 is constant. The fibers of any of the projections $X \to E_j$ are curves of genus 5, which are generically smooth. Taking $j = 1$, we get explicit equations for the fiber in the form

$$y_2^2 = x_2^4 + a_2 x_2^2 z_2^2 + z_2^4$$
$$y_3^2 = x_3^4 + a_3 x_3^2 z_3^2 + z_3^4$$
$$\xi_1 x_2 x_3 = c\zeta_1 z_2 z_3$$

where $(\xi_1 : \eta_1 : \zeta_1) \in E_1$ is the point we take the fiber over. If $\xi_1 = 0$ or $\zeta_1 = 0$, then the fiber splits into two of the curves described in Section 6.1 (taken twice). Otherwise, the third equation is equivalent to $(x_3 : z_3) = (c\zeta_1 z_2 : \xi_1 x_2)$. Using this in the second equation, we can reduce to the pair

$$y_2^2 = x_2^4 + a_2 x_2^2 z_2^2 + z_2^4$$
$$y_3^2 = \xi_1^4 x_2^4 + a_3 c^2 \xi_1^2 \zeta_1^2 x_2^2 z_2^2 + c^4 \zeta_1^4 z_2^4$$

of equations describing the fiber as a bidouble cover of $\mathbb{P}^1$.

The fiber degenerates if and only if the two quartic forms in (x_2, z_2) have common roots. This occurs exactly when their resultant

$$R = \left(c^4 \xi_1^4 \zeta_1^4 (a_2^2 + a_3^2) - c^2 \xi_1^2 \zeta_1^2 (\xi^4 + c^4 \zeta_1^4) a_2 a_3 + (\xi_1^4 - c^4 \zeta_1^4)^2 \right)^2$$

vanishes. Writing $\xi = \xi_1/(c\zeta_1)$, this is equivalent to

$$\xi^8 - a_2 a_3 \xi^6 + (a_2^2 + a_3^2 - 2)\xi^4 - a_2 a_3 \xi^2 + 1 = 0\,.$$

So generically (in terms of a_2 and a_3) this occurs for eight values of ξ, and then the two quartic forms have one pair of common roots, leading to a fiber that is a curve of geometric genus 3 with two nodes. But it is also possible that both quartics are proportional. This happens if and only if $a_2 = \pm a_3$ (and $\xi = \pm 1$ if $a_2 = a_3$, $\xi = \pm i$ if $a_2 = -a_3$). In that case, the fiber splits into two curves isomorphic to E_2 and E_3 that meet transversally in the four points where $y_2 = y_3 = 0$. The two values of ξ give rise to four fibers (note that the point $(\xi_1 : \eta_1 : \zeta_1)$ on E_1 cannot have $\eta_1 = 0$, since then X would contain a fixed point of γ), together containing eight copies of E_2 (or E_3). The action of Γ permutes these eight curves transitively. The stabilizer of a fiber swaps its two components and interchanges the four intersection points in pairs, so that the image of either component on S is a curve of geometric genus 1 with two nodes. Note that the existence of these curves does not depend on the value of c (as long as X is smooth).

If both $a_2 = a_3$ and $a_2 = -a_3$, so that $a_2 = a_3 = 0$ (which means that the curves have complex multiplication by $\mathbb{Z}[i]$), then we obtain two orbits of this kind, leading to two curves of geometric genus 1 on S. So in the extreme case $a_1 = a_2 = a_3 = 0$, we obtain the maximal number of six such curves on S. This situation is shown in Figure 3 in terms of the images of the curves in $\mathbb{P}^2$ with respect to Burniat's construction. In general, the table below gives the number of these curves (the subscripts are understood up to cyclic permutation).

condition	number of curves
$a_1^2 \neq a_2^2 \neq a_3^2 \neq a_1^2$	0
$0 \neq a_1^2 = a_2^2 \neq a_3^2$	1
$0 = a_1 = a_2 \neq a_3$	2
$0 \neq a_1^2 = a_2^2 = a_3^2$	3
$0 = a_1 = a_2 = a_3$	6

Note that the fibrations of X given by the projections to the E_j are stable under Γ. They induce fibrations $S \to \mathbb{P}^1$ with fibers of genus 3. In Burniat's construction, these fibrations are obtained by pulling back the pencils of lines through one of the three points on $\mathbb{P}^2$ that are blown up. Using this, it is easy to see that the fibers in X over the ramification points of $E_j \to \mathbb{P}^1$ map to smooth curves of genus 2 on S (taken twice).

Definition 6.2. We call the curves of geometric genus 1 on S arising in this way *curves of type I.*

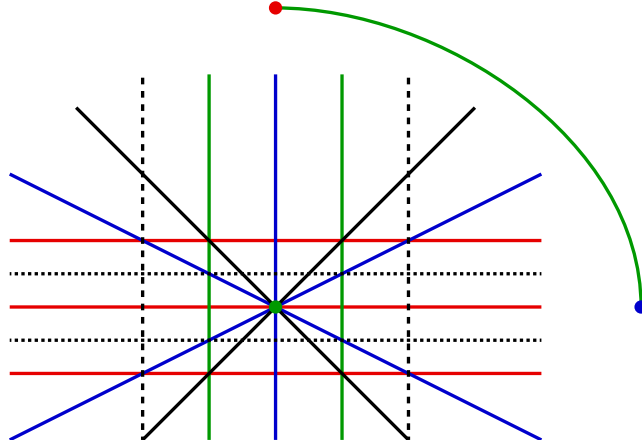See the left part of Figure 5 for how a curve of type I sits inside S.

FIGURE 3. Six curves of type I. They are the preimages in S of the six black lines. Here two of the blown-up points are at infinity.

We write N_1 for the subset of M given by $d = 0$ (so that two of the a_j agree up to sign, compare Theorem 4.2), which corresponds to surfaces admitting curves of type I.

6.3. **No ψ_j is constant.** With a slight abuse of notation we write x_j for the rational function x_j/z_j on E_j (and X). If $E \subset X$ is an elliptic curve and all ψ_j are non-constant, then each ψ_j has the form $\psi_j(P) = \varphi_j(P - Q_j)$, where $Q_j \in E$ is some point and $\varphi_j \colon E \to E_j$ is an isogeny. The condition that E is contained in X then means that

$$(x_1 \circ \psi_1) \cdot (x_2 \circ \psi_2) \cdot (x_3 \circ \psi_3) = c \,.$$

In particular, we must have

$$\tau_{Q_1} \varphi_1^* \big(\mathrm{div}(x_1) \big) + \tau_{Q_2} \varphi_2^* \big(\mathrm{div}(x_2) \big) + \tau_{Q_3} \varphi_3^* \big(\mathrm{div}(x_3) \big)$$
$$= \mathrm{div}(x_1 \circ \psi_1) + \mathrm{div}(x_2 \circ \psi_2) + \mathrm{div}(x_3 \circ \psi_3) = 0 \,,$$

where τ_Q denotes translation by Q.

It can be checked that the four points above 0 and ∞ on E_j form a principal homogeneous space for the 2-torsion subgroup $E_j[2]$. So, taking the coefficients mod 2, $\mathrm{div}(x_j) \equiv [2]^*(P_j)$ for some point $P_j \in E_j$, where $[2]$ denotes the multiplication-by-2 map. The relation above then implies that

$$\tau_{Q_1'} [2]^* K_1 + \tau_{Q_2'} [2]^* K_2 + \tau_{Q_3'} [2]^* K_3 \equiv 0 \bmod 2$$

with suitable points $Q'_j \in E$, where K_j denotes the formal sum of the points in the kernel of φ_j. This in turn is equivalent to

$$K_1 + \tau_{2Q'_2 - 2Q'_1} K_2 + \tau_{2Q'_3 - 2Q'_1} K_3 \equiv 0 \bmod 2\,.$$

In down-to-earth terms, this means that there are cosets of $\ker(\varphi_1)$ (which we can take to be $\ker(\varphi_1)$ itself), $\ker(\varphi_2)$ and $\ker(\varphi_3)$ such that every point of E is contained in either none or exactly two of these cosets. Since $0 \in \ker(\varphi_1)$, exactly one of the two other cosets must contain 0; we can assume that this is $\ker(\varphi_2)$. Then the condition is that the symmetric difference of $\ker(\varphi_1)$ and $\ker(\varphi_2)$ must be a coset of $\ker(\varphi_3)$.

Since $E \subset X \subset E_1 \times E_2 \times E_3$, the intersection of all three kernels is trivial. Now assume that $G = \ker(\varphi_1) \cap \ker(\varphi_2) \neq \{0\}$. Then the symmetric difference of $\ker(\varphi_1)$ and $\ker(\varphi_2)$ is a union of cosets of G, which implies that $G \subset \ker(\varphi_3)$, a contradiction. So $\ker(\varphi_1) \cap \ker(\varphi_2) = \{0\}$ (and the same holds for the other two intersections) and hence the symmetric difference is $\ker(\varphi_1) \cup \ker(\varphi_2) \setminus \{0\}$. Now if $\# \ker(\varphi_1) > 2$, then this symmetric difference contains two elements of $\ker(\varphi_1)$ whose difference then must be in $\ker(\varphi_1) \cap \ker(\varphi_3)$, which is impossible. So $\# \ker(\varphi_1) \leq 2$ and similarly $\# \ker(\varphi_2) \leq 2$. On the other hand, the kernels cannot both be trivial (then the symmetric difference would be empty and could not be a coset). So, up to cyclic permutation of the subscripts, this leaves two possible scenarios:

(1) $\# \ker(\varphi_1) = 2$ and $\ker(\varphi_2)$ and $\ker(\varphi_3)$ are trivial;
(2) $\ker(\varphi_1)$, $\ker(\varphi_2)$ and $\ker(\varphi_3)$ are the three subgroups of order 2 of E.

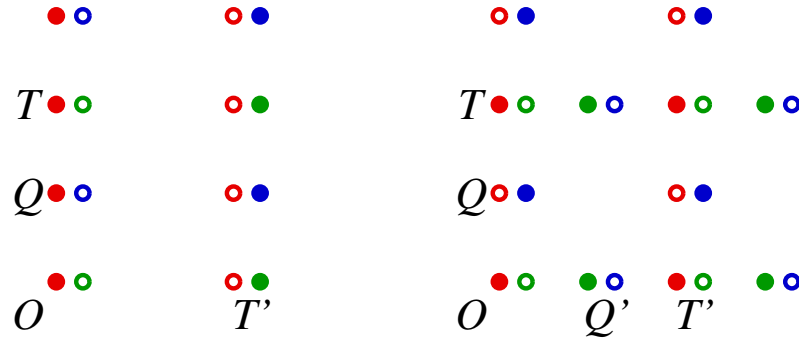We now have to check whether and how these possibilities can indeed be realized.

FIGURE 4. The possible arrangements of the divisors of $x_1 \circ \psi_1$ (red), $x_2 \circ \psi_1$ (green) and $x_3 \circ \psi_3$ (blue). On the left is Scenario (1), on the right Scenario (2). $\bullet$ denotes a zero, $\circ$ a pole of $x_j \circ \psi_j$.

In Scenario (1), E, E_2 and E_3 are isomorphic. We identify E_2 and E_3 with E via the isomorphisms ψ_2 and ψ_3. The points above 0 and ∞ on E_2 and E_3 then have to be distinct, and the two corresponding cosets of $E[2]$ must differ by a point Q of order 4 such that $\ker(\varphi_1) = \{0, 2Q\}$. In $\mathrm{div}(x_1 \circ \psi_1)$, points differing by $2Q$ then have the same sign, which implies that $T_2 = T_3 = 2Q =: T$ (recall that T_j is the point of order 2 on E_j such that adding T_j changes the sign of x_j). Let $T' \in E$ be a point of order 2 with $T' \neq T$. Then we can realize this scenario by taking

$$\psi_1 \colon E \to E_1 := E/\langle T\rangle, \quad \psi_2 = \mathrm{id}_E \colon E \to E_2 := E, \quad \psi_3 = \mathrm{id}_E \colon E \to E_3 := E$$

and choosing the x_j such that

$$\mathrm{div}(x_1 \circ \psi_1) = (O) + (Q) + (T) + (-Q) - (T') - (T' + Q) - (T' + T) - (T' - Q),$$
$$\mathrm{div}(x_2 \circ \psi_2) = (T') + (T + T') - (O) - (T),$$
$$\mathrm{div}(x_3 \circ \psi_3) = (T' + Q) + (T' - Q) - (Q) - (-Q).$$

See the left part of Figure 4.

More concretely, let

$$E \colon y^2 = x(x - 1)(x - \lambda^2),$$

which has a point $Q = (\xi, \eta) = (\lambda, i\lambda(1 - \lambda))$ of order 4 such that $T = 2Q = (0, 0)$. Then $E_1 = E/\langle T\rangle$ is

$$E_1 \colon y^2 = (x - 2\lambda)(x + 2\lambda)(x - (1 + \lambda^2));$$

with

$$x_1 = \frac{(1 + \lambda)y}{(x + 2\lambda)(x - (1 + \lambda^2))} \quad \text{and} \quad y_1 = \frac{x^2 - 4\lambda x + 4\lambda(1 - \lambda + \lambda^2)}{(x + 2\lambda)(x - (1 + \lambda^2))}$$

this gives the equation

$$y_1^2 = x_1^4 + 2\frac{1 - 6\lambda + \lambda^2}{(1 + \lambda)^2}x_1^2 + 1$$

for E_1. On $E_2 = E = E_3$, we take

$$x_2 = \frac{1}{\sqrt{1 - \lambda^2}}\frac{y}{x} \quad \text{and} \quad x_3 = \sqrt{-\frac{1 - \lambda}{1 + \lambda}}\frac{x + \lambda}{x - \lambda}.$$

This gives the following equations for E_2 and E_3:

$$y_2^2 = x_2^4 + 2\frac{1 + \lambda^2}{1 - \lambda^2}x_2^2 + 1, \qquad y_3^2 = x_3^4 + 2\frac{1 + \lambda^2}{1 - \lambda^2}x_3^2 + 1$$

with $y_2 = 1/(1 - \lambda^2) \cdot (x - \lambda^2/x)$ and $y_3 = 4i\lambda/(1 + \lambda) \cdot y/(x - \lambda)^2$. Since the isogeny $E \to E_1$ is given by $(x, y) \mapsto (x + \lambda^2/x, (1 - \lambda^2/x^2)y)$, we get indeed that

$$(x_1 \circ \psi_1)x_2 x_3 = \frac{(1 + \lambda)(x - \lambda)y}{(x + \lambda)(x - 1)(x - \lambda^2)} \cdot \frac{1}{\sqrt{1 - \lambda^2}} \frac{y}{x} \cdot \sqrt{-\frac{1 - \lambda}{1 + \lambda}} \frac{x + \lambda}{x - \lambda}$$

$$= \pm i \frac{y^2}{x(x - 1)(x - \lambda^2)} = \pm i$$

is constant. (The sign depends on the choice of the two square roots.)

We can read off the intersection multiplicities with the elliptic curves 'at infinity' of X from the divisors of the functions $x_j \circ \psi_j$: a point that occurs positively in $\mathrm{div}(x_j \circ \psi_j)$ and negatively in $\mathrm{div}(x_k \circ \psi_k)$ contributes one (transversal) intersection of E with a curve in $X \cap \{x_j = z_k = 0\}$ and therefore also contributes 1 to the intersection multiplicity of $\pi(E)$ with the image of these curves on S. In our case, we obtain intersection multiplicity 2 for

$$x_1 = z_2 = 0, \quad x_1 = z_3 = 0, \quad z_1 = x_2 = 0, \quad z_1 = x_3 = 0,$$

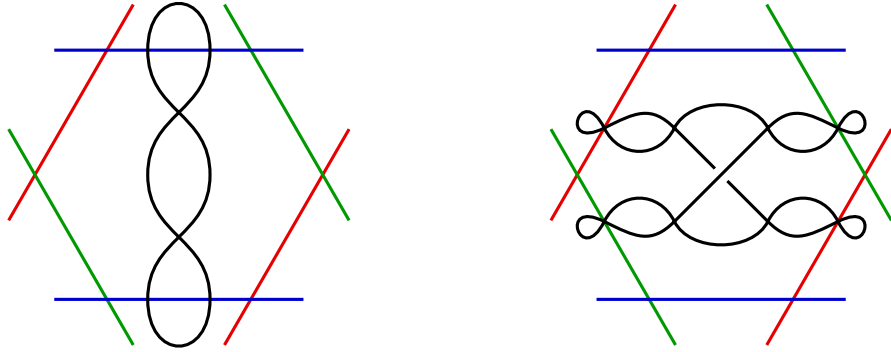and no intersection for $x_2 = z_3 = 0$ and for $z_2 = x_3 = 0$.

FIGURE 5. A curve of type I (left) and a curve of type II (right) sitting inside S.

The Γ-orbit of such a curve has size 8. In terms of points P on E, the action of Γ on E is as follows. We write $\psi = \psi_1$. Note that the automorphisms that negate the y-coordinate are given, respectively, by $P \mapsto \psi(Q) - P$, $P \mapsto T - P$, $P \mapsto -P$

on E_1, E_2, E_3.

	E_1	E_2	E_3	intersection
id	$\psi(P)$	P	P	same curve
γ_1	$\psi(P)$	$T+P$	$T-P$	$P = O, T, T', T+T'$
γ_2	$-\psi(P)$	P	$T+P$	none
γ_3	$\psi(Q)+\psi(P)$	$-P$	P	none
$\gamma_2\gamma_3$	$\psi(Q)-\psi(P)$	$-P$	$T+P$	none
$\gamma_1\gamma_3$	$\psi(Q)+\psi(P)$	$T-P$	$T-P$	$2P = \pm Q$ (8 points)
$\gamma_1\gamma_2$	$-\psi(P)$	$T+P$	$-P$	$P = Q, -Q, T'+Q, T'-Q$
γ	$\psi(Q)-\psi(P)$	$T-P$	$-P$	none

In the column labeled 'intersection', we list the points $P \in E$ that lie on the corresponding translate of E. The entries show that each curve in the orbit intersects two other curves in the orbit in four points each, which are on some of the curves at infinity, and intersects another curve in eight points. These intersection points are swapped in pairs by the element of Γ that interchanges the two intersecting curves. So the image of this orbit in S is a curve of geometric genus 1 with eight nodes (and hence arithmetic genus 9). Its image on $\mathbb{P}^2$ is a conic passing through exactly two of the blown-up points, which at these points is tangent to one of the lines in the branch locus, intersects the two lines joining the two points with the third at an intersection point with another branch line and is tangent to two further branch lines; see Figure 6, Left. It can be checked that there can be at most one such conic through a given pair of the points unless the two corresponding E_j have $a_j = 0$. In this case, there are either none or two such conics, see Figure 6, Right. (When $a_2 = a_3 = 0$, say, then $(x_1, y_1, x_2, y_2, x_3, y_3) \mapsto (-x_1, y_1, ix_2, y_2, ix_3, y_3)$ is an additional automorphism of X, which descends to an automorphism of S that swaps the two curves.)

If such a curve is to exist on S, two of the a_j must agree up to sign, and the third must satisfy a relation with the first two, which is (for $a_2 = a_3$ as above)

$$\left(a_1 - 2(a_2^2 - 3)\right)^2 = 4a_2^2(a_2^2 - 4).$$

In addition, we must have $c^2 = -1$ if $a_2 = a_3 \neq 0$, and $c^4 = 1$ otherwise. Of course, we can replace a_1 by $-a_1$; then $c^2 = 1$ if $a_2 = a_3 \neq 0$. This translates into (u_1, u_2, u_3, w) being on a curve (and $d = 0$, $v = 4$).

The locus in M of points whose associated surfaces contain curves of type II is a smooth rational curve of degree 6. One possible parameterization is

$$u_1 = -2\frac{t^3 - 4t^2 - 7t - 8}{t^2}$$

$$u_2 = \frac{(t-1)^2(t^3 - 10t^2 - 31t - 32)}{t^3}$$

$$u_3 = 4\frac{(t-1)^4(t+2)^2}{t^4}$$

$$d = 0$$

$$v = 4$$

$$w = -4\frac{(t-1)^2(t+2)}{t^2}$$

We write N_2 for this rational curve in M.

There are four values of λ such that all a_j are equal up to sign; this leads to $a_j = \pm\alpha$ with $\alpha = \frac{3}{8}(5 \pm \sqrt{-7})$ and

$$(u_1, u_2, u_3, d, v, w) = (3\alpha^2, 3\alpha^4, \alpha^6, 0, 4, 2\alpha^3).$$

In this case, E has complex multiplication by the order of discriminant -7, and the isogeny $E \to E_1$ is multiplication by $(1 \pm \sqrt{-7})/2$. So in this special situation, we can have three curves of this type.
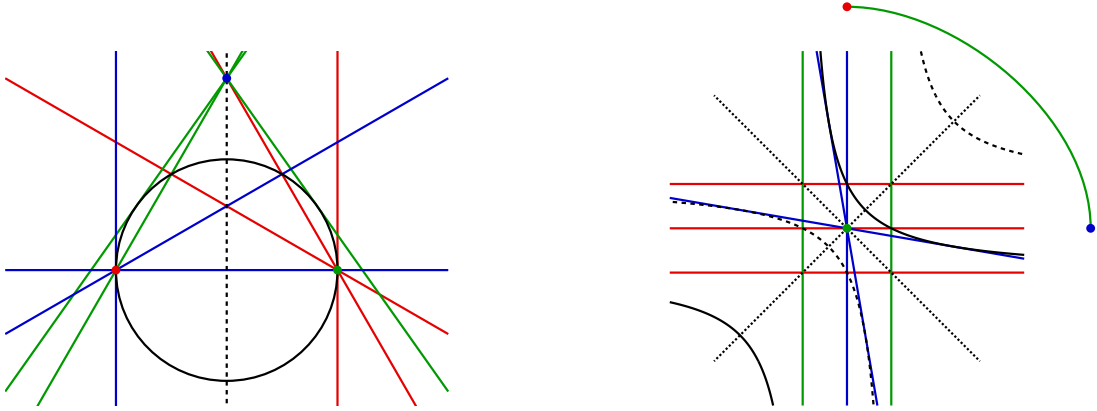

FIGURE 6. **Left:** A curve of type II. It is the preimage in S of the black conic. The dashed line leads to a curve of type I.
Right: When two of the curves have $a_j = 0$, then we can obtain two curves of type II (given by the solid and the dashed black conics) and two curves of type I (given by the two dotted lines).

Definition 6.3. We call the curves of geometric genus 1 on S arising as in Scenario (1) *curves of type II*.

See the right part of Figure 5 for how a curve of type II sits inside S.

For scenario (2), we consider E with two points Q, Q' of order 4 that span $E[4]$. We set $T = 2Q$, $T' = 2Q'$ and write $D = (O) + (T) + (T') + (T + T')$ for the divisor that is the sum of the 2-torsion points on E. Then essentially the only way to realize this scenario is via

$$\psi_1 \colon E \longrightarrow E_1 := E/\langle T \rangle \qquad \text{with} \quad \operatorname{div}(x_1 \circ \psi_1) = D - \tau_Q(D) \,,$$
$$\psi_2 \colon E \longrightarrow E_2 := E/\langle T' \rangle \qquad \text{with} \quad \operatorname{div}(x_2 \circ \psi_2) = \tau_{Q'}(D) - D \,,$$
$$\psi_3 \colon E \longrightarrow E_3 := E/\langle T + T' \rangle \quad \text{with} \quad \operatorname{div}(x_3 \circ \psi_3) = \tau_Q(D) - \tau_{Q'}(D) \,.$$

See the right part of Figure 4.

We are now going to show that this scenario is impossible. For this, note that the map negating the y-coordinate is given on E_1, E_2 and E_3 by $P \mapsto \psi_1(T') - P$, $P \mapsto \psi_2(T) - P$ and $P \mapsto -P$, respectively. Consider $P = Q + Q' \in E$. We have

$$P = \big(\psi_1(Q + Q'), \psi_2(Q + Q'), \psi_3(Q + Q')\big)$$
$$= \big(\psi_1(-Q + Q'), \psi_2(Q - Q'), \psi_3(-Q - Q')\big)$$
$$= \big(\psi_1(T') - \psi_1(P), \psi_2(T) - \psi_2(P), -\psi_3(P)\big)$$
$$= \gamma(P) \,,$$

so that γ would have P as a fixed point on X, which contradicts our assumptions. So this scenario is indeed impossible.

We summarize our findings.

Theorem 6.4. *Let S be a primary Burniat surface. Then S does not contain rational curves. Depending on the location of the moduli point of S in M, S contains the numbers of curves of geometric genus 1 at infinity, of type I and of type II as given in Table 1. The column labeled 'dim.' gives the dimension of the corresponding subset of the moduli space.*

In particular, the number of curves of geometric genus 1 on a primary Burniat surface can be 6, 7, 8, 9, 10 or 12.

In each case, $\operatorname{Aut}(S)$ acts transitively on the set of curves of type I and on the set of curves of type II.

Definition 6.5. For our purposes, we will say that a primary Burniat surface S is *generic*, if S does not contain curves of type I or II.

By Theorem 6.4, S is generic if and only if no two of the E_j are isomorphic as double covers of $\mathbb{P}^1$, which means that $d \neq 0$.

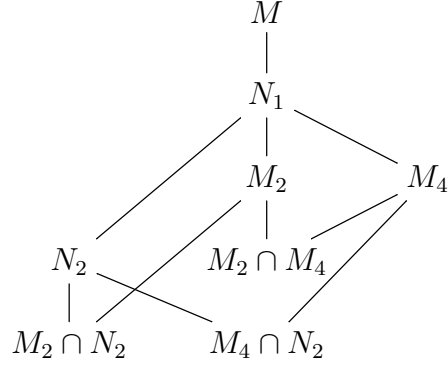| $\in$	$\notin$	∞	I	II	dim.
M	N_1	6	0	0	4
N_1	M_2, M_4, N_2	6	1	0	3
M_2	M_4, N_2	6	2	0	2
M_4	M_2, N_2	6	3	0	2
$M_2 \cap M_4$		6	6	0	1
N_2	M_2, M_4	6	1	1	1
$M_2 \cap N_2$		6	2	2	0
$M_4 \cap N_2$		6	3	3	0

TABLE 1. Curves of geometric genus 1 on S, see Theorem 6.4. We use the notations M_j, N_j as introduced in Sections 4 and 6. Each line applies to surfaces whose moduli point is in the subset of M given in the '$\in$' column, but not in any of the subsets given in the '$\notin$' column.
$M_2 \cap N_2$ is the single point $(u_1, u_2, u_3, d, v, w) = (36, 0, 0, 0, 4, 0)$ represented by $(a_1, a_2, a_3, c) = (0, 0, 6, 1)$, whereas $M_4 \cap N_2$ consists of the two points $(3\alpha^2, 3\alpha^4, \alpha^6, 0, 4, 2\alpha^3)$, where $4\alpha^2 - 15\alpha + 18 = 0$, represented by $(-\alpha, -\alpha, -\alpha, i)$.

7. Rational points on primary Burniat surfaces

In this section we will assume that everything is defined over $\mathbb{Q}$. Concretely, this means that the curves E_j are curves of genus 1 over $\mathbb{Q}$, the torsion points T_j are rational points on the Jacobian elliptic curves of the E_j, the functions x_j are defined over $\mathbb{Q}$, and $c \in \mathbb{Q}$. Then X and S are also defined over $\mathbb{Q}$, and it makes sense to study the set $S(\mathbb{Q})$ of rational points on S.

We recall that S contains six genus 1 curves at infinity; we will soon see that these curves always have rational points, so they are elliptic curves over $\mathbb{Q}$. If some of these curves have positive Mordell-Weil rank, then $S(\mathbb{Q})$ is infinite. The same is true when S contains curves of type I or II that are defined over $\mathbb{Q}$, have a rational point and positive Mordell-Weil rank.

Definition 7.1. Let S be a primary Burniat surface. We denote by S' the complement in S of the union of the curves of geometric genus 1 on S. Note that $S'(\mathbb{Q})$ is the set of sporadic rational points on S.

Theorem 7.2. *If S a primary Burniat surface, then the set $S'(\mathbb{Q})$ of sporadic rational points on S is finite.*
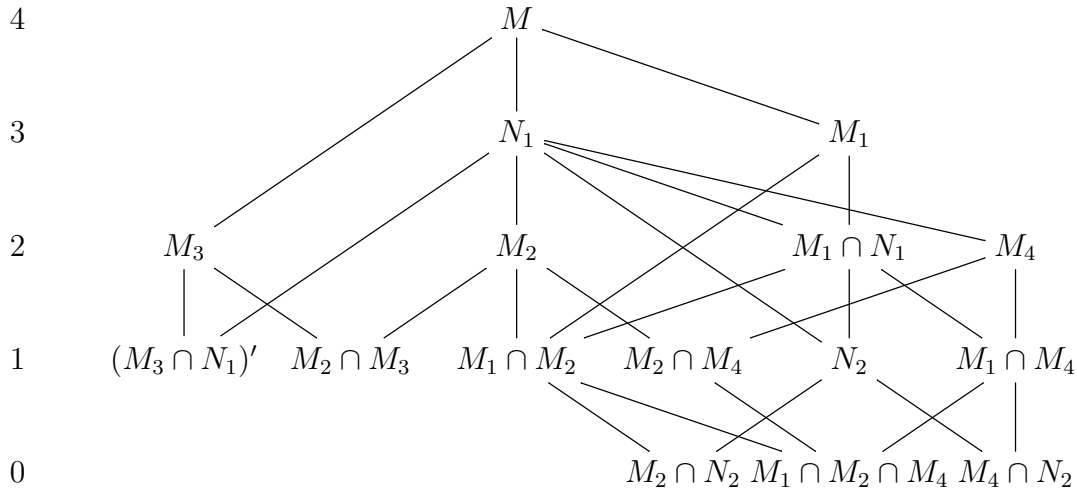
FIGURE 7. The combined stratification. Note that $M_3 \cap N_1$ is the disjoint union of $M_2 \cap M_3$ and $(M_3 \cap N_1)'$. The latter is represented by $a_1 = 0$, $a_2 = a_3$, $c^4 + 1 = 0$.

Proof. This is a special case of Theorem 2.3. We use the fact that the only low-genus curves on S are the curves of geometric genus 1 that we have excluded from S'. $\qquad\square$

In the remainder of this paper, we will be concerned with computing the set of sporadic rational points explicitly.

We now discuss how to modify the description given in Section 4 when we want to work over $\mathbb{Q}$ instead of over $\mathbb{C}$. We will restrict to the case that the three genus 1 curves E_1, E_2, E_3 whose product contains X are individually defined over $\mathbb{Q}$. Then the two commuting involutions on E_j used in the construction, one without fixed points and one with four fixed points, are also defined over $\mathbb{Q}$. Dividing by the latter, we see that E_j is (as before) a double cover of $\mathbb{P}^1$ ramified in four points, so it can be given by an equation of the form $y^2 = f(x)$ with a polynomial f of degree (at most) 4. The function that is invariant under the second involution and changes sign under the first must then be a function on $\mathbb{P}^1_x$; we can choose x to be that function. Then the first involution is $(x, y) \mapsto (-x, -y)$ (without the sign change on y, it would have fixed points), and the polynomial f must be even. So the curve E_j has (affine) equation

$$E_j \colon y_j^2 = r_j x_j^4 + s_j x_j^2 + t_j$$

with $r_j, s_j, t_j \in \mathbb{Q}$ and $(s_j^2 - 4r_j t_j)r_j t_j \neq 0$. Note that the parameter a_j in equations (4.1) is then given by $a_j^2 = s_j^2 / r_j t_j$. The surface X is given by $x_1 x_2 x_3 = c$

inside $E_1 \times E_2 \times E_3$. The moduli point on M is given by

$$u_1 = \frac{s_1^2}{r_1 t_1} + \frac{s_2^2}{r_2 t_2} + \frac{s_3^2}{r_3 t_3}, \; u_2 = \frac{s_1^2 s_2^2}{r_1 r_2 t_1 t_2} + \frac{s_2^2 s_3^2}{r_2 r_3 t_2 t_3} + \frac{s_3^2 s_1^2}{r_3 r_1 t_3 t_1}, \; u_3 = \frac{s_1^2 s_2^2 s_3^2}{r_1 r_2 r_3 t_1 t_2 t_3},$$

$$d = \left(\frac{s_1^2}{r_1 t_1} - \frac{s_2^2}{r_2 t_2} \right) \left(\frac{s_2^2}{r_2 t_2} - \frac{s_3^2}{r_3 t_3} \right) \left(\frac{s_3^2}{r_3 t_3} - \frac{s_1^2}{r_1 t_1} \right), \; v = \frac{r_1 r_2 r_3}{t_1 t_2 t_3} c^4 + 2 + \frac{t_1 t_2 t_3}{r_1 r_2 r_3} c^{-4}$$

$$\text{and} \quad w = s_1 s_2 s_3 \left(\frac{c^2}{t_1 t_2 t_3} + \frac{c^{-2}}{r_1 r_2 r_3} \right).$$

The generators of Γ act as follows.

$$\gamma_1 \colon (x_1, y_1, x_2, y_2, x_3, y_3) \longmapsto (x_1, y_1, -x_2, -y_2, -x_3, y_3)$$
$$\gamma_2 \colon (x_1, y_1, x_2, y_2, x_3, y_3) \longmapsto (-x_1, y_1, x_2, y_2, -x_3, -y_3)$$
$$\gamma_3 \colon (x_1, y_1, x_2, y_2, x_3, y_3) \longmapsto (-x_1, -y_1, -x_2, y_2, x_3, y_3).$$

Remark 7.3. One also obtains a primary Burniat surface defined over $\mathbb{Q}$, when the three curves E_j are defined over a cyclic cubic extension K of $\mathbb{Q}$ and form a Galois orbit (and $c \in \mathbb{Q}$). Then Γ is a $\mathbb{Q}$-group scheme that splits over K. In Burniat's construction, this means that the three points p_i are defined over K and permuted cyclically by the action of the Galois group of K over $\mathbb{Q}$. We do not look into this case further, or consider the question for which moduli points $p \in M(\mathbb{Q})$ a corresponding primary Burniat surface can be defined over $\mathbb{Q}$.

By Remark 2.4, every rational point on S lifts to one of finitely many twists $X_\xi \to S$ of $X \to S$. Our method for determining $S(\mathbb{Q})$ will be to first find the relevant twists X_ξ and then determine the set of rational points $X_\xi(\mathbb{Q})$ on each of them. The first step in this approach is to give an explicit description of the twists of $X \to S$. In general, these twists are parameterized by the Galois cohomology set $H^1(\mathbb{Q}, \Gamma)$; in our case, we obtain the Galois cohomology group (it is a group, because Γ is abelian)

$$H^1(\mathbb{Q}, \Gamma) \cong H^1(\mathbb{Q}, \mu_2)^3 \cong \left(\mathbb{Q}^\times / (\mathbb{Q}^\times)^2 \right)^3.$$

Its elements can be represented by triples (d_1, d_2, d_3) of squarefree integers. The corresponding twist X_ξ is then given by

$$(7.1) \qquad X_{(d_1, d_2, d_3)} \colon \begin{cases} d_3 y_1^2 = r_1 d_2^2 d_3^2 x_1^4 + s_1 d_2 d_3 x_1^2 + t_1 \\ d_1 y_2^2 = r_2 d_1^2 d_3^2 x_2^4 + s_2 d_1 d_3 x_2^2 + t_2 \\ d_2 y_3^2 = r_3 d_1^2 d_2^2 x_3^4 + s_3 d_1 d_2 x_3^2 + t_3 \\ d_1 d_2 d_3 x_1 x_2 x_3 = c. \end{cases}$$

We can restrict the prime divisors of the d_j to the set consisting of the prime 2 and the primes p of bad reduction, i.e., such that the reduction of X contains a fixed point of $\gamma = \gamma_1 \gamma_2 \gamma_3$. To make this explicit, we have to define a kind of

'discriminant' for X that vanishes if and only if X passes through one of the fixed points. This will be the case if and only if $c = x_1(P_1)x_2(P_2)x_3(P_3)$ for some $P_j \in E_j$ that are fixed points of $(x_j, y_j) \mapsto (x_j, -y_j)$, i.e., for points $P_j = (\alpha_j, 0)$ with $r_j\alpha_j^4 + s_j\alpha_j^2 + t_j = 0$. Equivalently, $c^2 = \beta_1\beta_2\beta_3$ with β_j a root of $r_jX^2 + s_jX + t_j$. We therefore look at

$$\prod_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm1} \left(8r_1r_2r_3c^2 - (-s_1 + \varepsilon_1\Delta_1)(-s_2 + \varepsilon_2\Delta_2)(-s_3 + \varepsilon_3\Delta_3)\right),$$

where $\Delta_j = \sqrt{s_j^2 - 4r_jt_j}$. We can divide this by $2^{24}r_1^4r_2^4r_3^4$ and obtain

$$\begin{aligned}
D = {} & r_1^4r_2^4r_3^4\,c^{16} + r_1^3r_2^3r_3^3s_1s_2s_3\,c^{14} + r_1^2r_2^2r_3^2(4\sigma_1 - 2\sigma_2 + \sigma_3)\,c^{12} \\
& + r_1r_2r_3s_1s_2s_3(-5\sigma_1 + \sigma_2)\,c^{10} + (6\sigma_1^2 - 4\sigma_1\sigma_2 - 2\sigma_1\sigma_3 + \sigma_1\sigma_4 + \sigma_2^2)\,c^8 \\
& + s_1s_2s_3t_1t_2t_3(-5\sigma_1 + \sigma_2)\,c^6 + t_1^2t_2^2t_3^2(4\sigma_1 - 2\sigma_2 + \sigma_3)\,c^4 \\
& + s_1s_2s_3t_1^3t_2^3t_3^3\,c^2 + t_1^4t_2^4t_3^4 .
\end{aligned}$$

Here we have set

$$\begin{aligned}
\sigma_1 &= r_1r_2r_3t_1t_2t_3 \\
\sigma_2 &= r_1r_2s_3^2t_1t_2 + r_1r_3s_2^2t_1t_3 + r_2r_3s_1^2t_2t_3 \\
\sigma_3 &= r_1s_2^2s_3^2t_1 + r_2s_1^2s_3^2t_2 + r_3s_1^2s_2^2t_3 \\
\sigma_4 &= s_1^2s_2^2s_3^2 .
\end{aligned}$$

We see that X does not pass through fixed points of γ and is smooth (and therefore S is smooth) if and only if $cD \neq 0$ and the curves E_j are smooth. The latter condition is

$$0 \neq \prod_{j=1}^{3}(r_jt_j(s_j^2 - 4r_jt_j)) = \sigma_1(\sigma_4 - 4\sigma_3 + 16\sigma_2 - 64\sigma_1).$$

We obtain the following result.

Lemma 7.4. *Assume that the r_j, s_j, t_j and c are integers satisfying*

$$r_jt_j(s_j^2 - 4r_jt_j) \neq 0 \quad \text{for all } j \qquad \text{and} \qquad c \neq 0, \quad D \neq 0.$$

Let d_1, d_2, d_3 be squarefree integers. Then the twist $X_{(d_1,d_2,d_3)}$ of $X \to S$ fails to have points over some completion of $\mathbb{Q}$ unless all the prime divisors of $d_1d_2d_3$ are prime divisors of $2cD\sigma_1(\sigma_4 - 4\sigma_3 + 16\sigma_2 - 64\sigma_1)$.

We now consider the genus 1 curves at infinity on X and on S. On X, a typical Γ-orbit of these is as follows. We fix a point at infinity on E_1 and a point with zero x-coordinate on E_2, say those with $y_1/x_1^2 = \sqrt{r_1}$ and $y_2 = \sqrt{t_2}$; call them P

and Q and the other points with the same x-coordinate $\bar{P}$ and $\bar{Q}$. Then the curve $\{P\} \times \{Q\} \times E_3$ is contained in X, and the generators of Γ act as follows.

$$\gamma_1 \colon (P, Q, R) \longmapsto (P, \bar{Q}, -R + T_3)$$
$$\gamma_2 \colon (P, Q, R) \longmapsto (P, Q, R + T_3)$$
$$\gamma_3 \colon (P, Q, R) \longmapsto (\bar{P}, Q, R)\,.$$

The orbit of the curve consists of four curves, the other three being $\{\bar{P}\} \times \{Q\} \times E_3$, $\{P\} \times \{\bar{Q}\} \times E_3$ and $\{\bar{P}\} \times \{\bar{Q}\} \times E_3$. The action splits into independent actions on $\{P, \bar{P}\}$ and on $\{Q, \bar{Q}\} \times E_3$. Invariants for the latter are given by x_3^2 and $\sqrt{t_2}x_3y_3$ (where $Q \mapsto \bar{Q}$ corresponds to $\sqrt{t_2} \mapsto -\sqrt{t_2}$). There is the relation

$$(\sqrt{t_2}x_3y_3)^2 = t_2x_3^2(r_3x_3^4 + s_3x_3^2 + t_3)\,,$$

which in terms of $Y = \sqrt{t_2}x_3y_3$ and $X = x_3^2$ can be written as

$$Y^2 = t_2X(r_3X^2 + s_3X + t_3)\,,$$

which is the quadratic twist by t_2 of the Jacobian elliptic curve of E_3. Switching the roles of zero and infinity, we get the r_2-twist of the same elliptic curve. We obtain the other curves by cyclic permutation of the indices. This gives us the quadratic twists by t_3 and by r_3 of the Jacobian elliptic curve of E_1 and the quadratic twists by t_1 and by r_1 of the Jacobian elliptic curve of E_2. These six curves are arranged in the form of a hexagon, with intersection points at the origin and the obvious (rational) point of order 2. We can now give the following refinement of Theorem 7.2:

Theorem 7.5. *Assume that E_1, E_2, E_3 are defined over $\mathbb{Q}$ and that T_1, T_2, T_3 are rational points. Then we always have $\#S(\mathbb{Q}) \geq 6$. $S'(\mathbb{Q})$ is finite, and $S(\mathbb{Q})$ is infinite if and only if at least one of the genus 1 curves on S has a rational point and positive rank.*

Proof. The first statement follows, since the six points where adjacent elliptic curves at infinity in S intersect (compare Figure 2) are always rational. For the second statement, we recall that $S'(\mathbb{Q})$ is finite by Theorem 7.2. Therefore $S(\mathbb{Q})$ is infinite if and only if one of the genus 1 curves on S has infinitely many points, i.e., it has a rational point and positive rank. $\square$

Remark 7.6. If we consider a primary Burniat surface defined over $\mathbb{Q}$ as described in Remark 7.3, then the absolute Galois group of $\mathbb{Q}$ acts on the six curves at infinity and therefore also on their intersection points by cyclic permutations of order 3. So in this case, these six intersection points will not be rational (they are defined over K).

Remark 7.7. The preceding theorem shows that S always has rational points (under the assumptions of the theorem). This is not true in general for its étale covering X. For example, we can take E_1 to be without rational points as for the

surface considered in Proposition 9.1 below. However, there will always be a twist $X' \to S$ of the covering such that X' has rational points.

8. Determining the set of rational points on the twists

This section is dedicated to the discussion how to determine the set $X_\xi(\mathbb{Q})$ of rational points on the relevant twists of X. For this, note first that X_ξ is of the same form as X, as shown by the defining equations (7.1). For the description of the available methods, it is therefore sufficient to deal with X itself.

We have the three projections $\psi_j \colon X \hookrightarrow E_1 \times E_2 \times E_3 \to E_j$ at our disposal. There are two cases.

8.1. **Some E_j fails to have rational points.** If we can show that $E_j(\mathbb{Q}) = \emptyset$ for some j, then clearly $X(\mathbb{Q}) = \emptyset$, too. To decide whether E_j has rational points, we can, on the one hand, try to find a rational point by a systematic search, and, on the other hand, try to prove that no rational point exists by performing a 'second descent' on the 2-covering E_j of J_j, its Jacobian elliptic curve. The latter can be done explicitly by computing the '2-Selmer set' of E_j, compare [BS09]; if this set is empty, then $E_j(\mathbb{Q}) = \emptyset$ as well. If necessary, a further descent step can be performed on the two-coverings of E_j, compare [Sta05].

8.2. **All E_j have rational points.** If we find some $P_j \in E_j(\mathbb{Q})$ for all j, then the E_j are elliptic curves, and we can try to determine the free abelian rank of the groups $E_j(\mathbb{Q})$. The standard tool for this is Cremona's mwrank program as described in [Cre97], whose functionality is included in Magma and SAGE. This is based on 2-descent. If this is not sufficient to determine the rank, then one can also perform 'higher descents', see for example [CFO$^+$08] and the references given there.

There are several cases, according to how many of the sets $E_j(\mathbb{Q})$ are finite.

All $E_j(\mathbb{Q})$ are finite. Then $(E_1 \times E_2 \times E_3)(\mathbb{Q})$ is finite, and we only have to check which of these finitely many points lie on X.

Two of the $E_j(\mathbb{Q})$ are finite. Say $E_1(\mathbb{Q})$ and $E_2(\mathbb{Q})$ are finite. Then we can easily determine the set of rational points on X by checking the finitely many fibers of $(\psi_1, \psi_2) \colon X \to E_1 \times E_2$ above rational points of $E_1 \times E_2$; note that these fibers are finite (of size two), hence it is easy to determine their rational points.

Exactly one of the $E_j(\mathbb{Q})$ is finite. Say $E_1(\mathbb{Q})$ is finite. In this case, we can at least reduce to the finitely many fibers of ψ_1 over rational points on E_1. These fibers are (generically) smooth curves of genus 5 contained in the product of the two other genus 1 curves (and defined by the relation that the product of the two x-coordinates is constant).

We then need a way of determining the set of rational points on a genus 5 curve C sitting in a product of two elliptic curves of positive rank. After making a transformation $x \leftarrow c'/x$ of the x-coordinate on one of the two genus 1 curves, the product condition becomes the condition that the two x-coordinates are equal. The curve C can then be written as a bidouble cover of $\mathbb{P}^1$, given by a pair of equations

$$u^2 = a_0 x^4 + a_1 x^2 z^2 + a_2 z^4, \qquad v^2 = b_0 x^4 + b_1 x^2 z^2 + b_2 z^4.$$

This curve is a double cover of the hyperelliptic curve

$$D \colon y^2 = (a_0 x^4 + a_1 x^2 z^2 + a_2 z^4)(b_0 x^4 + b_1 x^2 z^2 + b_2 z^4),$$

which is of genus 3 (unless the two factors have common roots), so the Jacobian of C splits up to isogeny into a product of three elliptic curves (with the Jacobians of E_1 and E_2 among them) and an abelian surface. More precisely, D covers the genus 1 curve

$$E \colon y^2 = (a_0 x'^2 + a_1 x' z' + a_2 z'^2)(b_0 x'^2 + b_1 x' z' + b_2 z'^2)$$

and the genus 2 curve

$$F \colon w^2 = x' z'(a_0 x'^2 + a_1 x' z' + a_2 z'^2)(b_0 x'^2 + b_1 x' z' + b_2 z'^2)$$

(where $(x' : z' : w) = (x^2 : z^2 : xyz)$). If the third elliptic curve has rank zero or the Jacobian of F has Mordell-Weil rank at most 1 (the rank can usually be determined by the methods of [Sto01]), then methods are available that in many cases will be able to determine $C(\mathbb{Q})$, see for example [BS10]. Even when the rank is 2 or larger, 'Elliptic curve Chabauty' methods might apply to F or also D, see [Bru03].

The possible degenerations of a fiber of the genus 5 fibration $X \to E_j$ are as follows.

(1) A curve of geometric genus 3 with two nodes. Then D and F above each degenerate to a curve of (geometric) genus 1, and we can hope that at least one of them has only finitely many rational points.
(2) Two curves of geometric genus 1 intersecting in four points. These map to curves of type I on S; we can try to find out whether they have finitely many or infinitely many rational points.

All $E_j(\mathbb{Q})$ are infinite. In this situation, there must be finitely many points $p_1, \ldots, p_r \in E_j(\mathbb{Q})$, such that the sporadic rational points of X are contained in the fibers over $p_1, \ldots, p_r$. But there are no methods available so far to determine these points, hence we cannot yet deal with this situation.

We end this section by taking a quick look at the situation when there are curves of type I or II on S and observe the following.

Remark 8.1. If a curve of type I or II on S has only finitely many rational points, it will not present any difficulties. Otherwise:

   (I) When there is a curve E of type I (defined over $\mathbb{Q}$) on S such that $E(\mathbb{Q})$ is infinite, then there will be some twist X_ξ of X such that E lifts to an elliptic curve E' on X_ξ with infinitely many rational points. Since E' is isomorphic to two of the $E_{j,\xi}$, we need the third $E_{j,\xi}$ to be of rank 0 for our approach to work.

 (II) When there is a curve E of type II (defined over $\mathbb{Q}$) on S such that $E(\mathbb{Q})$ is infinite, then there will again be some twist X_ξ of X such that E lifts to an elliptic curve E' on X_ξ with infinitely many rational points. In this case, E' is isogenous to all three $E_{j,\xi}$, so all three curves will have positive rank, and our current methods will necessarily fail.

9. Examples

We first give a number of examples in the generic case, when S does not have curves of types I or II.

As a first example, take

$$(9.1) \qquad E_1\colon y_1^2 = -x_1^4 - 1\,, \quad E_2\colon y_2^2 = x_2^4 - 1\,, \quad E_3\colon y_3^2 = x_3^4 + 1$$

and $c = 2$. All three curves have bad reduction only at 2, and $D = 50625 = 3^4 \cdot 5^4$, so the set of bad primes is $\{2, 3, 5\}$. There are $2^{3\cdot4} = 4096$ triples (d_1, d_2, d_3) to be checked. Of these, only $(1, -1, 1)$, $(1, -1, -1)$, $(2, -2, 2)$ and $(2, -2, -2)$ satisfy the condition that the twisted product of the E_j has a rational point. It turns out that all the twisted genus 1 curves that occur have only finitely many points. We can find them all and thence determine the sets $X_{(d_1,d_2,d_3)}(\mathbb{Q})$. It turns out that these sets are empty for the triples $(2, -2, 2)$ and $(2, -2, -2)$, and for the other two triples all these points map to points at infinity on S; they fall into five orbits each under the action of Γ on the two twists. This results in the following.

Proposition 9.1. *Let $X \subset E_1 \times E_2 \times E_3$ with E_j as in (9.1) be defined by $x_1x_2x_3 = 2$ and set $S = X/\Gamma$. Then $S'(\mathbb{Q}) = \emptyset$ and $\#S(\mathbb{Q}) = 10$.*

As our second example, we consider

$$(9.2) \quad E_1\colon y_1^2 = -x_1^4 - 1\,, \quad E_2\colon y_2^2 = x_2^4 + 2x_2^2 + 2\,, \quad E_3\colon y_3^2 = -2x_3^4 - 2x_3^2 + 1$$

and $c = 1$. The curves have bad reduction at most at 2 and 3, and $D = 256 = 2^8$, so the set of bad primes is $\{2, 3\}$. There are six triples (d_1, d_2, d_3) such that the corresponding twist of $E_1 \times E_2 \times E_3$ has rational points, namely $(1, -1, 1)$, $(1, -1, 2)$, $(1, -2, 2)$, $(-2, -1, 1)$, $(-2, -1, 2)$ and $(-2, -2, 2)$. For the third and the last, the twisted product has only finitely many rational points. The third gives no contribution (the rational points are not on the twist of X), whereas the last gives two rational points at infinity on S. For the remaining four triples, exactly one of the twisted curves has infinitely many rational points (rank 1). However, the other two curves have only rational points with $x_j \in \{0, \infty\}$, so all rational points on the corresponding twists of X map to points at infinity of S, of which there are now infinitely many (two of the six genus 1 curves at infinity on S have infinitely many rational points). We obtain the following.

Proposition 9.2. *Let $X \subset E_1 \times E_2 \times E_3$ with E_j as in (9.2) be defined by $x_1 x_2 x_3 = 1$ and set $S = X/\Gamma$. Then $S'(\mathbb{Q}) = \emptyset$ and $S(\mathbb{Q})$ is infinite.*

As our third example, we take

$$(9.3) \quad E_1: y_1^2 = 2x_1^4 + x_1^2 + 1, \quad E_2: y_2^2 = x_2^4 - x_2^2 + 1, \quad E_3: y_3^2 = x_3^4 - x_3^2 + 4$$

and $c = 1$. The curves have bad reduction at 2, 3, 5 and 7, and $D = 436 = 2^2 \cdot 109$, so that the set of bad primes is $\{2, 3, 5, 7, 109\}$. There are two triples (d_1, d_2, d_3) such that the corresponding twist of the product has rational points; they are $(1, 1, 1)$ and $(1, 1, 2)$. In both cases, two of the twisted curves have rank 1 and the third has finitely many rational points. For the second triple, these points all have $x \in \{0, \infty\}$, so that we do not obtain a contribution to $S'(\mathbb{Q})$. For the first triple, the twist of E_2 has eight rational points, four of which have $x \in \{0, \infty\}$ and the other four have $x \in \{\pm 1\}$. For each of these four points, the curve D is given by

$$D: y^2 = (2x^4 + x^2 + 1)(4x^4 - x^2 + 1).$$

The elliptic curve E covered by it has rank 1, but for the genus 2 curve

$$F: y^2 = x(2x^2 + x + 1)(4x^2 - x + 1)$$

that is also covered by D, one can show (using the algorithm of [Sto01]) that its Jacobian has Mordell-Weil rank 1. Then Chabauty's method combined with the Mordell-Weil sieve (see [BS10]) can be used to show that

$$F(\mathbb{Q}) = \{\infty, (0, 0), (1, 4), (1, -4)\}.$$

Note that the twist $(1, 1, 1)$ is the original surface X. Translating back to the coordinates on E_1, E_2, E_3, we see that all points on X must have $x_1, x_2, x_3 \in \{\pm 1\}$. Since each of the three curves has a pair of rational points with $x = 1$ and also with $x = -1$, this gives a total of 32 rational points on X mapping to S'. They fall into four Γ-orbits and therefore give rise to four rational points on S'. Since of the six elliptic curves at infinity on S, four have positive rank, we obtain the following.

Proposition 9.3. *Let $X \subset E_1 \times E_2 \times E_3$ with E_j as in (9.3) be defined by $x_1 x_2 x_3 = 1$ and set $S = X/\Gamma$. Then $S'(\mathbb{Q})$ consists of four points and $S(\mathbb{Q})$ is infinite.*

To conclude, we present an example where S contains a curve of type I with infinitely many rational points. We take

$$(9.4) \quad E_1 \colon y_1^2 = 2x_1^4 + x_1^2 + 1\,, \quad E_2 \colon y_2^2 = x_1^4 + x_2^2 + 2\,, \quad E_3 \colon y_3^2 = x_3^4 - x_3^2 + 1$$

and $c = 1$. The curves have bad reduction at 2, 3 and 7, and $D = 100 = 2^2 \cdot 5^2$, so that the set of bad primes is $\{2, 3, 5, 7\}$. There are four triples (d_1, d_2, d_3) such that the corresponding twist of the product has rational points; they are $(1, 1, 1)$, $(1, 1, 2)$, $(2, 1, 1)$ and $(2, 1, 2)$. For the twists corresponding to the last three, there is always one twisted E_j that has only finitely many rational points, all of which have $x \in \{0, \infty\}$, so we obtain no contribution outside the curves at infinity on S from these twists. For the first twist (which is X itself), we have that E_1 and E_2 both have rank 1 and E_3 has eight rational points, four of which have $x \notin \{0, \infty\}$. The fibers above these four points form a Γ-orbit, and each fiber splits as a union of two elliptic curves intersecting in four points, thus producing a type I curve on S. The curves on X are isomorphic to E_1 (and E_2); they therefore have infinitely many rational points, and the same is true for the type I curve on S they map to. Since there are no further contributions, we have the following result.

Proposition 9.4. *Let $X \subset E_1 \times E_2 \times E_3$ with E_j as in (9.4) be defined by $x_1 x_2 x_3 = 1$ and set $S = X/\Gamma$. Then S contains exactly one curve of type I, which has infinitely many rational points, and no curve of type II. Furthermore, three out of the six curves at infinity on S contain infinitely many rational points. There are no sporadic rational points on S.*

Proof. It remains to show that the type I curve detected above is the only extra low-genus curve on S. This follows from the discussion in Section 6, since the moduli point of S is given by $(u_1, u_2, u_3, d, v, w) = (2, \frac{5}{4}, \frac{1}{4}, 0, 4, -1)$, which is in N_1, but neither in M_2, M_4 nor in N_2. Also, one checks that three of the six curves at infinity have positive rank. □

References

[BC11] I. Bauer and F. Catanese, *Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces*, Classification of algebraic varieties, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 49–76, DOI 10.4171/007-1/3. MR2779467 (2012f:14074) ↑1, 3, 3, 4, 4, 4

[BC12] Ingrid Bauer and Fabrizio Catanese, *Inoue type manifolds and Inoue surfaces: a connected component of the moduli space of surfaces with $K^2 = 7$, $p_g = 0$*, Geometry and arithmetic, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 23–56, DOI 10.4171/119-1/2 (English, with English and Italian summaries). MR2987651 ↑2

[Bog77] F. A. Bogomolov, *Families of curves on a surface of general type*, Dokl. Akad. Nauk SSSR **236** (1977), no. 5, 1041–1044 (Russian). MR0457450 (56 #15655) ↑2

[BM86] W. D. Brownawell and D. W. Masser, *Vanishing sums in function fields*, Math. Proc. Cambridge Philos. Soc. **100** (1986), no. 3, 427–434, DOI 10.1017/S0305004100066184. MR857720 ↑2

[Bru03] Nils Bruin, *Chabauty methods using elliptic curves*, J. Reine Angew. Math. **562** (2003), 27–49, DOI 10.1515/crll.2003.076. MR2011330 (2004j:11051) ↑8.2

[BS09] Nils Bruin and Michael Stoll, *Two-cover descent on hyperelliptic curves*, Math. Comp. **78** (2009), no. 268, 2347–2370, DOI 10.1090/S0025-5718-09-02255-8. MR2521292 (2010e:11059) ↑8.1

[BS10] _____, *The Mordell-Weil sieve: proving non-existence of rational points on curves*, LMS J. Comput. Math. **13** (2010), 272–306, DOI 10.1112/S1461157009000187. MR2685127 (2011j:11118) ↑8.2, 9

[Bur66] Pol Burniat, *Sur les surfaces de genre $P_{12} > 1$*, Ann. Mat. Pura Appl. (4) **71** (1966), 1–24 (French). MR0206810 (34 #6626) ↑1, 3

[CW32] C. Chevalley and A. Weil, *Un théorème d'arithmétique sur les courbes algébriques.*, C. R. Acad. Sci., Paris **195** (1932), 570–572 (French). ZMATH 58.0182.04 ↑2

[Cre97] J. E. Cremona, *Algorithms for modular elliptic curves*, 2nd ed., Cambridge University Press, Cambridge, 1997. MR1628193 (99e:11068) ↑8.2

[CFO⁺08] J. E. Cremona, T. A. Fisher, C. O'Neil, D. Simon, and M. Stoll, *Explicit n-descent on elliptic curves. I. Algebra*, J. Reine Angew. Math. **615** (2008), 121–155, DOI 10.1515/CRELLE.2008.012. MR2384334 (2009g:11067) ↑8.2

[Fal83] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), no. 3, 349–366 (German). Erratum in: Invent. Math. **75** (1984), 381. MR718935 (85g:11026a) ↑1

[Fal94] Gerd Faltings, *The general case of S. Lang's conjecture*, Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991), Perspect. Math., vol. 15, Academic Press, San Diego, CA, 1994, pp. 175–182. MR1307396 (95m:11061) ↑2, 2

[HS00] Marc Hindry and Joseph H. Silverman, *Diophantine geometry*, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, New York, 2000. An introduction. MR1745599 (2001e:11058) ↑2

[Ino94] Masahisa Inoue, *Some new surfaces of general type*, Tokyo J. Math. **17** (1994), no. 2, 295–319, DOI 10.3836/tjm/1270127954. MR1305801 (95j:14048) ↑1, 3, 3.3

[Kaw80] Yujiro Kawamata, *On Bloch's conjecture*, Invent. Math. **57** (1980), no. 1, 97–100, DOI 10.1007/BF01389820. MR564186 (81j:32030) ↑2

[Lan86] Serge Lang, *Hyperbolic and Diophantine analysis*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), no. 2, 159–205, DOI 10.1090/S0273-0979-1986-15426-1. MR828820 (87h:32051) ↑2

[Lu10] Steven S. Y. Lu, *On surfaces of general type with maximal Albanese dimension*, J. Reine Angew. Math. **641** (2010), 163–175, DOI 10.1515/CRELLE.2010.032. MR2643929 (2011h:14047) ↑2

[MLP01] Margarida Mendes Lopes and Rita Pardini, *A connected component of the moduli space of surfaces with $p_g = 0$*, Topology **40** (2001), no. 5, 977–991, DOI 10.1016/S0040-9383(00)00004-5. MR1860538 ↑1

[Miy08] Yoichi Miyaoka, *The orbibundle Miyaoka-Yau-Sakai inequality and an effective Bogomolov-McQuillan theorem*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 403–417, DOI 10.2977/prims/1210167331. MR2426352 (2009g:14043) ↑2

[Ser97] Jean-Pierre Serre, *Lectures on the Mordell-Weil theorem*, 3rd ed., Aspects of Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1997. Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt; With a foreword by Brown and Serre. MR1757192 (2000m:11049) ↑2

[Sko01] Alexei Skorobogatov, *Torsors and rational points*, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press, Cambridge, 2001. MR1845760 (2002d:14032) ↑2.4

[Sta05] Sebastian Stamminger, *Explicit 8-descent on elliptic curves*, PhD thesis, International University Bremen, 2005. ↑8.1

[Sto01] Michael Stoll, *Implementing 2-descent for Jacobians of hyperelliptic curves*, Acta Arith. **98** (2001), no. 3, 245–277, DOI 10.4064/aa98-3-4. MR1829626 (2002b:11089) ↑8.2, 9

[Sto07] ――――, *Finite descent obstructions and rational points on curves*, Algebra Number Theory **1** (2007), no. 4, 349–391, DOI 10.2140/ant.2007.1.349. MR2368954 (2008i:11086) ↑2

[Vol85] José Felipe Voloch, *Diagonal equations over function fields*, Bol. Soc. Brasil. Mat. **16** (1985), no. 2, 29–39, DOI 10.1007/BF02584799. MR847114 ↑2

Mathematisches Institut, Universität Bayreuth, 95440 Bayreuth, Germany.

E-mail address: Ingrid.Bauer@uni-bayreuth.de

Mathematisches Institut, Universität Bayreuth, 95440 Bayreuth, Germany.

E-mail address: Michael.Stoll@uni-bayreuth.de