ERRATA:
FINITE DESCENT OBSTRUCTIONS AND RATIONAL POINTS ON CURVES

MICHAEL STOLL

In this note, we correct some mistakes in the paper

> M. Stoll: *Finite descent obstructions and rational points on curves,*
> Algebra & Number Theory **1**:4 (2007), 349–391.

1. Complex infinite places

The first correction is that in the definition of $X(\mathbb{A}_k)_\bullet$ and its variants, the factors corresponding to *complex* infinite places must be left out completely. This does not make a difference when X is geometrically connected (as in this case, the relevant factors are reduced to one point), but it does when X is not geometrically connected. The point is that over $\mathbb{C}$ any point on X lifts to any torsor, and so no information whatsoever can be gained at complex places.

This is relevant for the statement and proof of Lemma 5.10 (and therefore for Propositions 5.11 and 5.12 as well, and at all places where we reduce without loss of generality to X geometrically connected). Indeed, the proof breaks down when v or w is a complex place, since there are no non-squares in k_v (or k_w) in this case. In fact, the statement is false with the original definition of $X(\mathbb{A}_k)_\bullet$, since for example with $X = \operatorname{Spec} k \coprod \operatorname{Spec} k = \{P_1, P_2\}$ and $k = \mathbb{Q}(i)$, the set $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ contains the element (Q_v) that has $Q_v = P_1$ for all finite v and $Q_v = P_2$ for the infinite place v.

2. Connectedness of torsors

The second mistake is more serious. Laurent Moret-Bailly pointed out to me that the claim on the bottom of page 364 that (Y_0, G_0) is an X-torsor, where (Y, G) is an X-torsor, Y_0 is a k-component of Y, and G_0 is the stabilizer of Y_0, is incorrect. He gives the simple example $X = \operatorname{Spec} \mathbb{Q}$, $G = \mu_3$, Y the trivial torsor. Here one of the two $\mathbb{Q}$-components of Y is not a torsor under the stabilizer (which is trivial). What happens here is that the stabilizer may fail to act transitively on the fibers of $Y_0 \to X$.

The fallacious argument is mainly used in conjunction with Lemma 5.5 to conclude that $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \neq \emptyset$ implies that it suffices to consider geometrically connected torsors, in particular for Lemma 5.7 (1). However, this application can be saved. We strengthen Lemma 5.5 as follows.

Date: March 29, 2017.

Lemma 5.5′. *Assume that X is geometrically connected. If there is a torsor $(Y, G) \in \mathcal{C}ov(X)$ such that no twist Y_ξ has a geometric component defined over k, then $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \emptyset$. The analogous statement holds for the solvable variant.*

Proof. Assume that $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \neq \emptyset$. By Prop. 5.17 (whose proof depends only on Prop. 5.9, which in turn depends only on the basic properties in Lemma 5.3), there is a twist Y_ξ such that $Y_\xi(\mathbb{A}_k)_\bullet^{\text{f-cov}} \neq \emptyset$. Without loss of generality, $Y_\xi = Y$. Write $Y = Y_1 \coprod \ldots \coprod Y_n$ as a disjoint union of k-connected schemes. By Prop 5.11 (which depends only on Prop. 5.9 and Lemma 5.10), $Y_j(\mathbb{A}_k)_\bullet^{\text{f-cov}} \neq \emptyset$ for some $1 \leq j \leq n$. In particular, $Y_j(\mathbb{A}_k)_\bullet \neq \emptyset$, which implies that Y_j is geometrically connected, contradicting the assumption. The same proof works for $X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$ if G is solvable. □

The relevant fact here is that $(P_v) \in X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ does not just lift to Y, but to a k-component of Y. It would be good to be able to extend this statement (which is weaker than Prop. 5.17) to the abelian case as well; this would show that all results remain valid also in the abelian context.

We now give a more precise description of the necessary changes.

(1) In Lemma 5.7 (1), the part referring to abelian coverings must be removed.
(2) In Lemma 5.8 (3), the claim in the abelian case is conditional on $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \neq \emptyset$ (in this case there is a cofinal family of coverings by Lemma 5.7 (1), which is then a cofinal family of abelian coverings as well).
(3) In Propositions 5.14 and 5.15, we can only claim the 'easy' inclusion "$\subset$" in the abelian case.
(4) Proposition 5.16 remains valid, since its proof uses only the easy part of Proposition 5.15.
(5) The statement "$A(\mathbb{A}_k)_\bullet^{\text{f-cov}} = A(\mathbb{A}_k)_\bullet^{\text{f-sol}} = A(\mathbb{A}_k)_\bullet^{\text{f-ab}}$" near the bottom of page 373 remains valid, since $A(k) \neq \emptyset$ and so $A(\mathbb{A}_k)_\bullet^{\text{f-cov}} \neq \emptyset$ as well.
(6) If X is a principal homogeneous space for an abelian variety A, then we can write X as an n-covering of A for some n: $X \xrightarrow{\pi} A$. Consider the family $\mathcal{F} \subset \mathcal{A}b(X)$ of all m-coverings of A (for some m divisible by n) factoring through π. We have the implication $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \neq \emptyset \Longrightarrow \xi \in Ш(k, A)_{\text{div}}$. In addition, $\xi \in Ш(k, A)_{\text{div}}$ implies that $\mathcal{F}$ is a cofinal family of (abelian) coverings of X, from which (together with $\xi \in Ш(k, A)_{\text{div}}$) we can deduce that $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \neq \emptyset$. If $\mathcal{F}$ is a cofinal family of coverings, then we also get that $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet^{\text{f-sol}} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. These equalities also hold (trivially) when $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \emptyset$. So the statements on page 374 between the proof of Corollary 6.3 and the statement of Theorem 6.4 remain valid (and therefore the statements relating to $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ for curves C that rely on them remain valid as well).

3. Proposition 8.5

The proof of Proposition 8.5 is slightly incorrect. The claim that "P is in the image of $Z(\mathbb{A}_k)_\bullet$ in $C(\mathbb{A}_k)_\bullet$" may fail at real infinite places (thanks to David Corwin for pointing this out to me). However, to apply Theorem 8.2 we only need that $P_v \in Z(k_v)$ for a set of places of density 1, so we can do without the assumption that $P_v \in Z(k_v)$ for the finitely many infinite places v.

4. Typos

Here is a list of (harmless) typographical errors in the paper.

1. Page 350, third line from the bottom. ‘follows it’ should read ‘that follows it’.
2. Page 351, second line. ‘namely’ should read ‘that is’, and the preceding comma and the comma after ‘zero’ in the next line should be removed.
3. Page 359, line 9. The comma before ‘that’ should be removed.
4. Page 362, third line from the bottom. The comma before ‘and’ should be removed.

Mathematisches Institut, Universität Bayreuth, 95440 Bayreuth, Germany.

E-mail address: Michael.Stoll@uni-bayreuth.de