

Designs and codes in regular semilattices

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Classical combinatorial designs

- ▶ Let V be a finite set of size v .
- ▶ $\binom{V}{k}$ denotes the set of all k -element subsets of V .
(Notation motivated by $\#\binom{V}{k} = \binom{v}{k}$.)
- ▶ **Definition.**
 $D \subseteq \binom{V}{k}$ is a t -(v, k, λ) (block) design if

$$\#\{B \in D \mid T \subseteq B\} = \lambda \quad \text{for all } T \in \binom{V}{t}$$

- ▶ elements of D : **blocks**.
- ▶ t : **strength** of D .
- ▶ λ : **index** of D .
- ▶ Important (and hard) special case $\lambda = 1$:
 D **Steiner system**.

Example

A 2-(6, 3, 2) design (strength 2, index 2) is given by

$$V = \{1, 2, 3, 4, 5, 6\}.$$

$$D = \{\{1, 2, 3\}, \{1, 2, 5\}, \{1, 3, 6\}, \{1, 4, 5\}, \{1, 4, 6\}, \\ \{2, 3, 4\}, \{2, 4, 6\}, \{2, 5, 6\}, \{3, 4, 5\}, \{3, 5, 6\}\}.$$

Verification.

- ▶ $\{1, 2\}$ is contained in 2 blocks: $\{1, 2, 3\}, \{1, 2, 5\}$
- ▶ $\{2, 5\}$ is contained in 2 blocks: $\{1, 2, 5\}, \{2, 5, 6\}$
- ▶ etc.

Example (cont.)

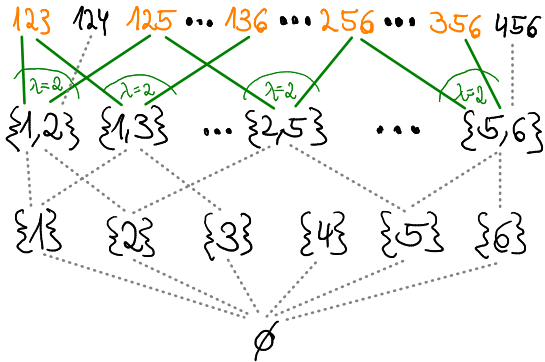
Poset view of the 2 -(6, 3, 2) design:

$$\Omega = X_3 = \binom{V}{3}$$

$$X_2 = \binom{V}{2}$$

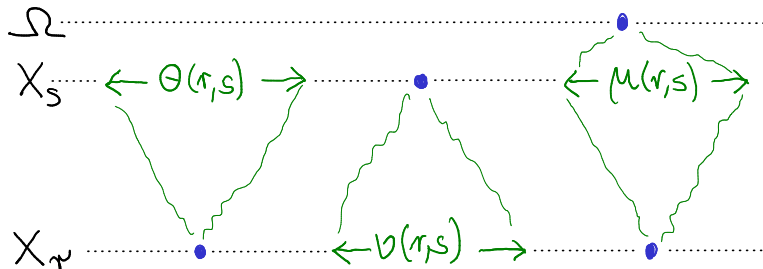
$$X_1 = \binom{V}{1}$$

$$X_0 = \binom{V}{0}$$



Generalized setting (Delsarte 1976)

- ▶ Let $(X, \leq)$ be a non-empty finite poset.
- ▶ $\wedge$ -semilattice: Any $a, b \in X$ have unique infimum $a \wedge b$.
- ▶ Minimum element $\perp$.
- ▶ Height function $h(x) = \ell(\perp, x)$.
- ▶ Levels $X_r := \{x \in X \mid h(x) = r\}$.
- ▶ $n := \max_{x \in X} h(x)$.
- ▶ Top level $\Omega := X_n$ plays a distinguished role.
- ▶ Three regularity properties: θ , ν , and μ -regularity.



Regularity properties

- ▶ $\theta(r, s) := \#\{x \in X_s \mid a \leq x\}$ is constant over all $a \in X_r$.
Mnemonic: **top**
- ▶ $\nu(r, s) := \#\{y \in X_r \mid y \leq b\}$ is constant over all $b \in X_s$.
Mnemonic: **nether, nadir**
- ▶ $\mu(r, s) := \#\{x \in X_s \mid a \leq x \leq b\}$
is constant over all $(a, b) \in X_r \times \Omega$ with $a \leq b$.
Mnemonic: **middle**

Remark

- ▶ In Delsarte 1976, additionally π -regularity.
 $\rightsquigarrow$ association scheme on Ω .
- ▶ Our setting: In general **no association scheme** on Ω .

Designs and codes in regular semilattices

- **Definition.** $D \subseteq \Omega$ is a **t -design** of **index λ** if

$$\#\{B \in D \mid T \leq B\} = \lambda \quad \text{for all } T \in X_t.$$

- For $\lambda = 1$: D is a **t -Steiner system**.
- **Definition.** $C \subseteq \Omega$, $\#C \geq 2$ is a **d -code** if

$$h(B \wedge B') \leq n - d$$

for all $B, B' \in C$ with $B \neq B'$.

- **Theorem (Singleton bound).**
For any d -code C ,

$$\#C \leq \frac{\theta(0, n)}{\theta(t, n)}.$$

where $t = n - d + 1$.

Equality $\iff C$ is **t -Steiner system**.

Example (Classical regular semilattices (Delsarte 1976))

- ▶ **Johnson** semilattice $J(v, n)$
subset lattice $(V, \subseteq)$, truncated at size n .
 - ▶ $h = \#$.
 - ▶ **designs**: classical designs.
 - ▶ **Steiner systems**: classical Steiner systems.
 - ▶ **codes**: constant weight codes.
- ▶ **q -Johnson** semilattice $J_q(v, n)$
 q -analog of the above $\rightsquigarrow$
subspace lattice $(V, \subseteq)$, truncated at dimension n .
 - ▶ $h = \dim$.
 - ▶ **designs**: subspace designs.
 - ▶ **Steiner systems**: q -analog Steiner systems.
 - ▶ **codes**: constant dimension subspace codes.

Example (Classical regular semilattices (Delsarte 1976))

- ▶ **Hamming** semilattice $H(m, n)$
poset of partial maps $N \rightarrow M$ (with $\#N = n$ and $\#M = m$).
 - ▶ $h = \# \text{domain}$.
 - ▶ Top level Ω consists of (total) maps $N \rightarrow M$.
 Ω corresponds to Hamming space, where
 N set of positions, M alphabet.
 - ▶ **codes**: block codes in the Hamming metric.
 - ▶ **Steiner systems**: MDS codes.
 - ▶ **designs**: orthogonal arrays.
- ▶ **q -Hamming** semilattice $H_q(m, n)$
 q -analog of the above $\rightsquigarrow$
poset of partial *linear* maps $\mathbb{F}_q^n \rightarrow \mathbb{F}_q^m$.
 - ▶ Top level Ω corresponds to matrix space $\mathbb{F}_q^{m \times n}$.
 - ▶ **codes**: matrix codes in the rank metric.
 - ▶ **Steiner systems**: MRD codes.
 - ▶ **designs**: — (not much studied).

► **Question.**

Which statements about classical designs extend to our poset setting?

— **Candidates:**

Statements formulated in terms of the poset.

- **Lemma (Suda 2012).** Let D be a t -design of index λ on Ω . Then for all $s \in \{0, \dots, t\}$, D is an s -design of index

$$\lambda_s := \frac{\theta(s, n)}{\theta(t, n)} \cdot \lambda.$$

- **Definition.** Let $D \subseteq \Omega$.

For $A \in X$, define the **intersection number**

$$\alpha_r(A) := \#\{B \in D \mid h(A \wedge B) = r\}.$$

► Theorem (K., Klawuhn 2026).

Let $D \subseteq \Omega$ be a t -design.

Mendelsohn equations. For every $r \in \{0, \dots, t\}$,

$$\sum_{s=r}^n \nu(r, s) \alpha_s(A) = \nu(r, h(A)) \lambda_r.$$

Köhler parametrization. For every $r \in \{0, \dots, t\}$,

$$\alpha_r(A) = \sum_{s=r}^t \nu'(r, s) \nu(s, h(A)) \lambda_s - \sum_{u=t+1}^n \left(\sum_{s=r}^t \nu'(r, s) \nu(s, u) \right) \alpha_u(A),$$

where $(\nu) = (\nu(r, s))_{r,s}$ (unitriangular matrix)
and $\nu'(r, s)$ entries of $(\nu)^{-1}$.

► Proof (Idea). Double counting; Gauss elimination.

Corollary (K., Klawuhn 2026).

Let $D \subseteq \Omega$ be a t -Steiner system. Then, for each block $B \in D$, the **block intersection distribution** at B is given by

$$\alpha_r(B) = \begin{cases} \sum_{s=r}^{t-1} \nu'(r, s) \nu(s, n) \left(\frac{\theta(s, n)}{\theta(t, n)} - 1 \right) & \text{if } 0 \leq r < t, \\ 0 & \text{if } t \leq r < n, \\ 1 & \text{if } r = n. \end{cases}$$

In particular, $\alpha_r(B)$ is constant over all blocks B .

Proof (idea).

- ▶ t -Steiner system property forces $\alpha_t(B) = \alpha_{t+1}(B) = \dots = \alpha_{n-1}(B) = 0$ and $\alpha_n(B) = 1$.
- ▶ Plug these values into the Köhler parametrization.



Application

- ▶ **Hamming** $\rightsquigarrow$ local distance distribution of **MDS codes**.
 - ▶ for linear codes: **Assmus-Mattson-Turyn 1965**,
Kasami-Lin-Peterson 1966, **Forney 1966**, **Goethals 1969**.
 - ▶ for unrestricted codes: **Marguinaud 1970**,
Heise-Quattrocchi 1983.
- ▶ **q -Hamming** $\rightsquigarrow$ local distance distribution of **MRD codes**.
 - ▶ averaged distribution: **Delsarte 1978**.
 - ▶ for linear codes: **Gabidulin 1985**,
Dumas-Gow-McGuire-Sheekey 2010, **Ravagnani 2016**.
 - ▶ for unrestricted codes: **Ravagnani 2018**.
- ▶ **Johnson** $\rightsquigarrow$ block intersection dist. of **Steiner systems**.
 - ▶ uniqueness: **Mendelsohn 1970**.
 - ▶ formula: **Goethals 1970**.
- ▶ **q -Johnson** $\rightsquigarrow$ block intersection dist. of **q -Steiner systems**.
 - ▶ likely **new**.

Evaluation of our approach

- ▶ Strongest variant of the statements:
 - ▶ For **individual codewords/blocks**, no averaging.
 - ▶ For **unrestricted codes**: not only linear or additive ones.
- ▶ **Unified** setting and proof.
- ▶ Identifies the **natural** underlying double-counting objects.
(Eigenvalue-based approaches typically involve more complicated expressions.)

Let $D \subseteq \Omega$.

► **Next Goal.**

Investigate **averaged intersection distribution** $(\bar{\alpha}_0, \dots, \bar{\alpha}_n)$ of D :

$$\bar{\alpha}_r := \frac{1}{\#D} \sum_{B \in D} \alpha_r(B).$$

(Essentially: Delsarte's **inner distribution** of Ω .)

► **Tool: Alternative characterization of designs.**

For $r \leq s$, define the **inclusion matrix** W^{rs} on the index set $X_r \times X_s$ by

$$W_{xy}^{rs} := \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

Then: $D \subseteq \Omega$ is a t -design $\iff W^{tn} \chi_D = \lambda \mathbf{1}$.

Theorem (K., Klawuhn 2026).

Let $D \subseteq \Omega$ and $(\bar{\alpha}_0, \dots, \bar{\alpha}_n)$ averaged intersection distribution.

(a) For all $r \in \{0, \dots, n\}$,

$$\sum_{s=r}^n \nu(r, s) \bar{\alpha}_s \geq \nu(r, n) \frac{\theta(r, n)}{\theta(0, n)} \cdot \#D.$$

(b) D is a t -design $\iff$ the inequality in (a) for $r = t$ is sharp.

Proof (Idea).

- Part (a). Apply Cauchy–Schwarz to $\langle W^r \chi_D, \mathbf{1} \rangle$.
- Part (b). Equality criterion of Cauchy–Schwarz.
+ alternative characterization of t -design.



Theorem (K., Klawuhn 2026), repeated.

Let $D \subseteq \Omega$ and $(\bar{\alpha}_0, \dots, \bar{\alpha}_n)$ its avg. intersection distribution.

(a) For all $t \in \{0, \dots, n\}$,

$$\sum_{s=t}^n \nu(t, s) \bar{\alpha}_s \geq \nu(t, n) \frac{\theta(t, n)}{\theta(0, n)} \cdot \#D.$$

(b) D is a t -design $\iff$ the t -th inequality in (a) is sharp.

Remarks

- ▶ Apparently new, even for classical designs.
- ▶ Advantages of Part (b) over Delsarte's algebraic characterization $\hat{a}_1 = \dots = \hat{a}_t = 0$ via dual distribution:
 - ▶ Single linear equation on the inner distribution (instead of t).
 - ▶ Combinatorially transparent.
 - ▶ Works even in settings without an association scheme.
- ▶ Clearly, $\bar{\alpha}_j \geq 0$.
- ▶ Regard $(x_0, \dots, x_n) = (\bar{\alpha}_0, \dots, \bar{\alpha}_n) \in \mathbb{R}^{n+1}$ as unknowns
 $\rightsquigarrow$ linear programming (= LP) bound for t -designs!

Theorem (K., Klawuhn 2026).

Let $D \subseteq \Omega$ be a non-empty t -design. Then:

$$\begin{aligned} \#D &\geq \text{optimum of the LP} \\ \text{minimize } &x_0 + \dots + x_n, \end{aligned}$$

with variables $x_0, \dots, x_n \in \mathbb{R}$, subject to

$$\sum_{i=0}^n \nu(r, i) x_i = \nu(r, n) \frac{\theta(r, n)}{\theta(0, n)} (x_0 + \dots + x_n) \quad (0 \leq r \leq t),$$

$$x_n = 1,$$

$$x_i \geq 0 \quad (0 \leq i \leq n),$$

$$\sum_{i=0}^n \nu(r, i) x_i \geq \nu(r, n) \frac{\theta(r, n)}{\theta(0, n)} (x_0 + \dots + x_n) \quad (t < r \leq n)$$

Comparison with Delsarte's LP bound

- ▶ We can show:

If $(X, \leq)$ is π -regular,

Delsarte's LP bound implies our LP bound.

$\implies$ Delsarte's LP bound is potentially stronger.

But. . .

- ▶ In many (but not all) concrete cases:
both LP bounds coincide.
- ▶ Our bound works even when
 - ▶ there is no association scheme on Ω , or
 - ▶ the eigenvalues of the association scheme
are unknown or too complicated.

Thank you!

First part: <https://arxiv.org/abs/2608.14437>

Second part: work in progress.

Slides will be uploaded at

<https://mathe2.uni-bayreuth.de/michaelk/>