

On codes and designs

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Outline

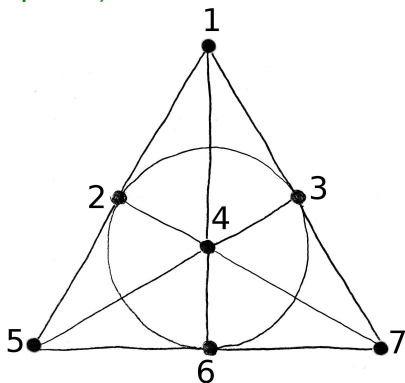
Introduction

Designs in regular semilattices

A new 3 -($42, 6, 1$) design

- ▶ Let V be a finite set of size v .
- ▶ $\binom{V}{k}$ denotes the set of all k -element subsets of V .
(Notation motivated by $\#\binom{V}{k} = \binom{v}{k}$.)
- ▶ **Definition.**
 $D \subseteq \binom{V}{k}$ is called a t - (v, k, λ) (block) design if each $T \in \binom{V}{t}$ lies in exactly λ elements of D .
- ▶ elements of D : **blocks**.
- ▶ t : **strength** of D .
- ▶ λ : **index** of D .
- ▶ In the important (and hard) special case $\lambda = 1$:
 D is called a **Steiner system**.

Example (Fano plane)



$$V = \{1, 2, 3, 4, 5, 6, 7\}$$

$$D = \{\{1, 2, 5\}, \{1, 4, 6\}, \{1, 3, 7\}, \{2, 3, 6\}, \\ \{2, 4, 7\}, \{3, 4, 5\}, \{5, 6, 7\}\}$$

Fano plane D is a 2-(7, 3, 1) design.

Connections between codes and designs

- ▶ **Assmus–Mattson theorem.**

Under certain conditions (essentially: large minimum distance and few dual weights), the codewords of fixed weight in a linear code form a t -design.

In particular: The two extended Golay codes carry 5-designs, including the small and the large Witt designs (which are Steiner systems).

- ▶ The blocks of certain designs can be used as parity-check equations for **majority-logic decoding**.
- ▶ Further connections \rightsquigarrow this talk.

Association schemes

Definition of (symmetric) association scheme

- ▶ Ω finite non-empty set.
- ▶ $\mathcal{R} = (R_0, \dots, R_n)$ partition of $\Omega \times \Omega$.

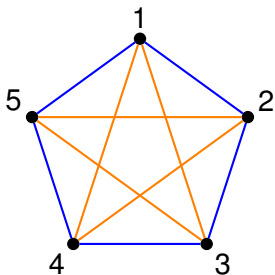
Think of each R_i as a

- ▶ relation on Ω ,
- ▶ graph on vertex set Ω .
- ▶ \mathcal{R} is (symmetric) n -class association scheme on Ω if
 - ▶ (R1) R_0 is the equality relation.
 - ▶ (R2) Each R_i is symmetric.
 - ▶ (R3) Regularity. For all $i, j, k \in \{0, \dots, n\}$,

$$p_{ij}^k := \#\{z \in \Omega \mid (x, z) \in R_i \text{ and } (z, y) \in R_j\}$$

is constant over all $(x, y) \in R_k$.

Example (Pentagon scheme)



$$\Omega = \{1, 2, 3, 4, 5\}$$

$$R_0 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$$

$$R_1 = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1), \\ (2, 1), (3, 2), (4, 3), (5, 4), (1, 5)\}$$

$$R_2 = \{(1, 3), (2, 4), (3, 5), (4, 1), (5, 2), \\ (3, 1), (4, 2), (5, 3), (1, 5), (2, 5)\}$$

$$p_{11}^1 = 0 \quad p_{12}^2 = 1 \text{ etc.}$$

Standard schemes in combinatorics (1)

Johnson scheme $J(v, n)$

- ▶ V set of size $v \geq 2n$
- ▶ $\Omega = \binom{V}{n}$
- ▶ $x R_i y \iff \#(x \cap y) = n - i.$

Natural ambient scheme for **block designs**.

Hamming scheme $H(q, n)$

- ▶ Alphabet Q of size q
- ▶ $\Omega = Q^n$
- ▶ $\mathbf{x} R_i \mathbf{y} \iff d_{\text{Ham}}(\mathbf{x}, \mathbf{y}) = i$

Natural ambient scheme for **block codes**.

Standard schemes in combinatorics (2)

Idea of q -analogs in combinatorics

Replace subsets by \mathbb{F}_q -subspaces!

q -Johnson (Grassmann) scheme $J_q(v, n)$

q -analog of Johnson scheme $J(v, n) \rightsquigarrow$

- ▶ V an \mathbb{F}_q -vector space of dimension $v \geq 2n$
- ▶ $\Omega = \begin{bmatrix} V \\ n \end{bmatrix}_q$ (\mathbb{F}_q -subspaces of dimension n)
- ▶ $x R_i y \iff \dim(x \cap y) = n - i$

Natural ambient scheme for **subspace designs**.

q -Hamming (bilinear forms) scheme $H_q(m, n)$

- ▶ $n \leq m$
- ▶ $\Omega = \mathbb{F}_q^{m \times n}$
- ▶ $x R_i y \iff \text{rk}(x - y) = i$

Natural ambient scheme for **rank-distance codes**.

Not an obvious q -analog of the Hamming scheme. *Stay tuned...*

Bose–Mesner algebra

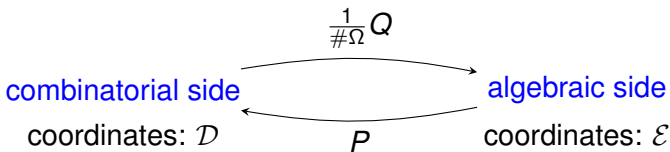
- ▶ Let $D_i \in \mathbb{C}^{\Omega \times \Omega}$ be adjacency matrix of R_i .
- ▶ $\mathcal{D} := \{D_0, \dots, D_n\}$.
- ▶ $\mathcal{A} := \langle \mathcal{D} \rangle_{\mathbb{C}}$ Bose–Mesner algebra.
- ▶ Properties:
 - ▶ closed under multiplication \implies indeed an algebra.
 - ▶ commutative.
- ▶ \implies set $\mathcal{E} = \{E_0, \dots, E_n\}$ of primitive idempotents is a \mathbb{C} -basis of \mathcal{A} .
- ▶ Henceforth: Assume fixed order

$$\mathcal{D} = (D_0, \dots, D_n) \quad \text{and} \quad \mathcal{E} = (E_0, \dots, E_n),$$

typically coming from a metric.

Eigenmatrices

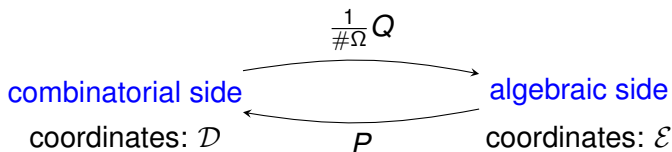
- ▶ Two distinguished bases:
 - ▶ **primal** basis \mathcal{D} (adjacency mat., **combinatorial side** of \mathcal{A})
 - ▶ **dual** basis \mathcal{E} (prim. idempotents, **algebraic side** of \mathcal{A})
- ▶ Change of basis $\mathcal{D} \rightarrow \mathcal{E} \rightsquigarrow$ coordinate change matrix P .
- ▶ Change of basis $\mathcal{E} \rightarrow \mathcal{D} \rightsquigarrow$ coordinate change matrix $\frac{1}{\#\Omega} Q$.
- ▶ P and Q (**first and second**) **eigenmatrices** of \mathcal{A} .
- ▶ On transformation level (opposite to coordinate change):



Example

The pentagon scheme has the eigenmatrices

$$P = Q = \begin{pmatrix} 1 & 2 & 2 \\ 1 & \frac{\sqrt{5}-1}{2} & -\frac{\sqrt{5}+1}{2} \\ 1 & -\frac{-\sqrt{5}+1}{2} & \frac{\sqrt{5}-1}{2} \end{pmatrix}.$$



Association schemes in combinatorics

- ▶ Spirit: Interplay of the two sides.
- ▶ Source of deep combinatorial results.
- ▶ Through involvement of algebraic side:
Often without transparent combinatorial explanation.

Delsarte theory (1)

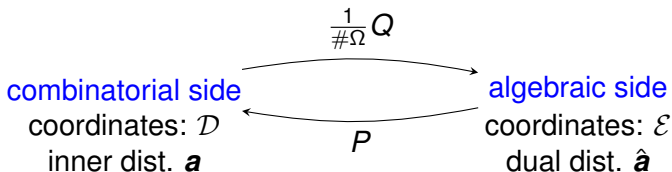
- ▶ Let $Y \subseteq \Omega$ be non-empty.
- ▶ **Inner distribution** of Y : $\mathbf{a} = (a_0, \dots, a_n)$ where

$$a_i = \frac{1}{\#Y} \cdot \#\{(x, y) \in Y \times Y \mid x R_i y\}.$$

Idea. average number of pairs in Y at “distance” i .

- ▶ Inner distribution lives at the combinatorial side.
- ▶ Transform to algebraic side \rightsquigarrow **dual distribution** of Y :

$$\hat{\mathbf{a}} = (\hat{a}_0, \dots, \hat{a}_n) := Q\mathbf{a}.$$



- ▶ **Delsarte 1973**: $\hat{a}_i \geq 0$ (innocent but powerful!)

Delsarte theory (2)

- ▶ $C \subseteq \Omega$ is a **d -code** in \mathcal{R} if $a_1 = a_2 = \dots = a_{d-1} = 0$.
- ▶ Clearly, d -codes in **Hamming scheme** $H(q, n)$ are **block codes** with minimum distance $\geq d$.
- ▶ \rightsquigarrow **Delsarte's linear programming bound for codes.**
Let $C \subseteq \Omega$ be a non-empty d -code. Then:

$\#C \leq$ optimum of the LP

$$\text{maximize } x_0 + \dots + x_n,$$

with variables $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$, subject to

$$x_0 = 1,$$

$$x_i = 0 \quad \text{for all } i \in \{1, \dots, d-1\},$$

$$x_i \geq 0 \quad \text{for all } i \in \{d, \dots, n\},$$

$$\hat{a}_k \geq 0 \quad \rightsquigarrow \quad \sum_{i=0}^n Q_{ki} x_i \geq 0 \quad \text{for all } k \in \{0, \dots, n\}.$$

Delsarte theory (3)

- ▶ $D \subseteq \Omega$ is a **t -design** in \mathcal{R} if $\hat{a}_1 = \hat{a}_2 = \dots = \hat{a}_t = 0$.
- ▶ **Delsarte 1973**. t -designs in the Johnson scheme are the same as combinatorial t -designs. (algebraic “miracle” ...)
- ▶ \rightsquigarrow **Delsarte’s linear programming bound for designs**.

Let $D \subseteq \Omega$ be a non-empty t -design. Then:

$\#D \geq$ optimum of the LP

$$\text{minimize } x_0 + \dots + x_n,$$

with variables $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$, subject to

$$\hat{a}_k = 0 \quad \rightsquigarrow \quad \sum_{i=0}^n Q_{ki} x_i = 0 \quad \text{for all } k \in \{1, \dots, t\},$$

$$x_0 = 1,$$

$$x_i \geq 0 \quad \text{for all } i \in \{1, \dots, n\},$$

$$\hat{a}_k \geq 0 \quad \rightsquigarrow \quad \sum_{i=0}^n Q_{ki} x_i \geq 0 \quad \text{for all } k \in \{t+1, \dots, n\}.$$

Remark on LP-bound

- ▶ Hamming and Johnson schemes: Can be **very strong**.
- ▶ But: Harder to evaluate; combinatorially less transparent.
- ▶ For q -Johnson, q -Hamming: typically weak.

Remark on Steiner systems

- ▶ In classical association schemes:
Steiner systems coincide
with certain classes of **extremal codes**.
- ▶ Johnson scheme: **perfect constant-weight** codes
- ▶ q -Johnson: **perfect constant-dimension** subspace code
- ▶ Hamming scheme: **MDS** codes
- ▶ q -Hamming scheme: **MRD** codes

Outline

Introduction

Designs in regular semilattices

A new 3 -($42, 6, 1$) design

Designs in regular semilattices

- ▶ Joint work with [Lukas Klawuhn](#)

Motivation

- ▶ **Observation.**

Inner distribution of a block design
= averaged **block intersection** distribution
([Mendelsohn](#), early 1970s).

- ▶ Associated Mendelsohn equations:
Meaning in association scheme framework?
- ▶ Follow [Delsarte 1976](#):
Designs in general \wedge -semilattices. Idea:
 - ▶ Association scheme does not “know”
about **intersections** of elements.
 - ▶ \rightsquigarrow Add \wedge -semilattice (X, \leq) beneath Ω .

Setting

- ▶ Let (X, \leq) be a non-empty finite graded \wedge -semilattice.
- ▶ Minimum element \perp .
- ▶ Height function h .
- ▶ Levels $X_i := \{x \in X \mid h(x) = i\}$.
- ▶ Top level $\Omega := X_n$ plays a distinguished role.
- ▶ Assume the following regularity properties for all r, s :
- ▶ $\theta(r, s) := \#\{x \in X_s \mid a \leq x\}$ is constant over all $a \in X_r$.
Mnemonic: **top**
- ▶ $\nu(r, s) := \#\{y \in X_r \mid y \leq b\}$ is constant over all $b \in X_s$.
Mnemonic: **nether**
- ▶ $\mu(r, s) := \#\{x \in X_s \mid a \leq x \leq b\}$
is constant over all $(a, b) \in X_r \times \Omega$ with $a \leq b$.
Mnemonic: **middle**

Theorem (Delsarte 1976).

Assume in addition that (X, \leq) is π -regular:

For all $r, s, t \in \{0, \dots, m\}$ for which there exists a pair $(a, b) \in (X_s \times \Omega)$ with $a \wedge b \in X_r$,

$$\pi(r, s, t) := \#\{(x, y) \in X_t \times \Omega \mid a \leq y \text{ and } x \leq y \text{ and } x \leq b\}$$

is constant over all such pairs $(a, b) \in (X_s \times \Omega)$.

Then the non-empty relations

$$x R_i y \iff h(x \wedge y) = n - i$$

form an association scheme on Ω .

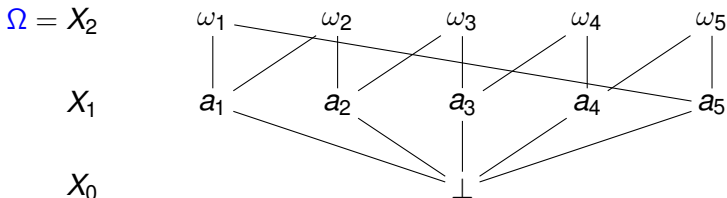
Remark

In this talk, we do not assume π -regularity.

Hence, in general, we do not have an association scheme on Ω .

Example (Pentagon scheme.)

- ▶ The pentagon scheme arises as the top level Ω of the following \wedge -semilattice.



- ▶ **Note.**
 - ▶ This \wedge -semilattice is **not** π -regular.
 - ▶ But Ω is still an association scheme!

Example

All four classical association schemes fit into this framework, i. e. arise as the top level Ω of a suitable \wedge -semilattice.

(with $x R_i y \iff h(x \wedge y) = n - i$.)

▶ **Johnson $J(v, n)$**

subset lattice (V, \subseteq) , truncated at size n .

▶ **q -Johnson $J_q(v, n)$**

q -analog of the above \rightsquigarrow

subspace lattice (V, \subseteq) , truncated at dimension n .

▶ **Hamming $H(q, n)$**

▶ $X =$ set of maps $f : A_f \rightarrow Q$ with $A_f \subseteq V$.

▶ $f \leq g \iff A_f \subseteq A_g$ and $g|_{A_f} = f$.

▶ $\rightsquigarrow h(f) = \#A_f$.

▶ **q -Hamming: $H_q(m, n)$**

q -analog of the above!! \rightsquigarrow

▶ $X =$ set of **linear** maps $f : A_f \rightarrow \mathbb{F}_q^m$ with **subspace** $A_f \subseteq V$.

▶ $f \leq g$: as before

▶ $\rightsquigarrow h(f) = \dim(A_f)$

- **Definition.** $D \subseteq \Omega$ is a t -design of index λ if

$$\#\{B \in D \mid x \leq B\} = \lambda \quad \text{for all } x \in X_t$$

- **Alternative characterization.**

For $s \leq t$, define the inclusion matrix W^{st} on the index set $X_s \times X_t$ by

$$W_{xy}^{st} := \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

Then: D is a t -design $\iff W^{tm} \chi_D = \lambda \mathbf{1}$.

- **Lemma (Suda 2012).** Let D be a t -design of index λ . Then for all $r \in \{0, \dots, t\}$, D is an r -design of index

$$\lambda_r := \frac{\theta(r, m)}{\theta(t, m)} \cdot \lambda.$$

- ▶ **Definition.** For $a \in X$, define the **intersection number**

$$\alpha_r(a) := \#\{x \in D \mid h(a \wedge x) = r\}.$$

- ▶ **Theorem (K., Klawuhn 2026).**

Mendelsohn equations. For all $r \in \{0, \dots, t\}$,

$$\sum_{s=r}^n \nu(r, s) \alpha_s(a) = \nu(r, h(a)) \lambda_r.$$

Kähler parametrization. For all $r \in \{0, \dots, t\}$,

$$\alpha_r(a) = \sum_{s=r}^t \nu'(r, s) \nu(s, h(a)) \lambda_s - \sum_{u=t+1}^n \left(\sum_{s=r}^t \nu'(r, s) \nu(s, u) \right) \alpha_u(a),$$

where $(\nu) = (\nu(r, s))_{r,s}$ (triangular mat. with unit diagonal) and $\nu'(r, s)$ entries of $(\nu)^{-1}$.

- ▶ **Proof (Idea).** Double counting; Gauss elimination.

Theorem (K., Klawuhn 2026).

Let $D \subseteq \Omega$ be a t -Steiner system. Then, for each block $B \in D$, the **block intersection distribution** at B is given by

$$\alpha_r(B) = \begin{cases} \sum_{s=r}^{t-1} \nu'(r, s) \nu(s, n) \left(\frac{\theta(s, n)}{\theta(t, n)} - 1 \right) & \text{if } 0 \leq r < t, \\ 0 & \text{if } t \leq r < n, \\ 1 & \text{if } r = n. \end{cases}$$

In particular, $\alpha_r(B)$ is constant over all blocks B .

Proof (idea).

- ▶ t -Steiner system property forces $\alpha_t(B) = \alpha_{t+1}(B) = \dots = \alpha_{n-1}(B) = 0$ and $\alpha_n(B) = 1$.
- ▶ Plug these values into the Köhler parametrization.



Application

The four classical schemes give the **distance distributions** of

- ▶ MDS codes ([Forney 1965](#))
- ▶ MRD codes
 - ▶ [Delsarte 1978](#), averaged version
 - ▶ [Ravagnani 2016](#), for linear codes, via dual code.
- ▶ perfect constant-weight codes ([Mendelsohn 1970](#))
- ▶ perfect constant-dimension subspace codes (**likely new**)

Evaluation of our approach

- ▶ Strongest variant of the statements:
 - ▶ For **individual codewords**, no averaging.
 - ▶ For **unrestricted codes**: not only linear or additive.
- ▶ **unified** proof.
- ▶ Identifies the **natural** underlying double-counting objects. (Delsarte's eigenmatrix approach yields more involved expressions in the proof.)

Theorem (K., Klawuhn 2026).

Let $D \subseteq \Omega$ and $(\bar{\alpha}_0, \dots, \bar{\alpha}_n)$ its **averaged intersection distribution**, i. e.

$$\bar{\alpha}_r := \frac{1}{\#D} \sum_{x \in D} \alpha_r(x).$$

(a) For all $t \in \{0, \dots, n\}$,

$$\sum_{s=t}^n \nu(t, s) \bar{\alpha}_s \geq \nu(t, n) \frac{\theta(t, n)}{\theta(0, n)} \cdot \#D.$$

(b) D is a t -design \iff the t -th inequality in (a) is sharp.

Proof (Idea).

- ▶ Use alternative characterization of t -design.
- ▶ Apply Cauchy–Schwarz to $\langle W^{tn} \chi_D, \mathbf{1} \rangle$.



Theorem (K., Klawuhn 2026), repeated.

Let $D \subseteq \Omega$ and $(\bar{\alpha}_0, \dots, \bar{\alpha}_n)$ its avg. intersection distribution.

(a) For all $t \in \{0, \dots, n\}$,

$$\sum_{s=t}^n \nu(t, s) \bar{\alpha}_s \geq \nu(t, n) \frac{\theta(t, n)}{\theta(0, n)} \cdot \#D.$$

(b) D is a t -design \iff the t -th inequality in (a) is sharp.

Remarks

- ▶ Essentially: average intersection distribution
= Delsarte's inner distribution.
- ▶ Advantages of Part (b)
over Delsarte's t -design characterization $\hat{a}_1 = \dots = \hat{a}_t = 0$:
 - ▶ **Single** equation on the inner distribution (instead of t).
 - ▶ Works even without an association scheme.
- ▶ Clearly, $\bar{\alpha}_j \geq 0$.
 \implies Part (a) yields a bound of linear programming type!

Theorem (K., Klawuhn 2026).

Let $D \subseteq \Omega$ be a non-empty t -design. Then:

$$\begin{aligned} \#D &\geq \text{optimum of the LP} \\ \text{minimize } & x_0 + \dots + x_n, \end{aligned}$$

with variables $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$, subject to

$$\sum_{i=0}^n \nu(r, \mathbf{e}_i) x_i = \nu(r, n) \frac{\theta(r, n)}{\theta(0, n)} (x_0 + \dots + x_n) \quad (0 \leq r \leq t),$$

$$x_0 = 1,$$

$$x_i \geq 0 \quad (0 \leq i \leq n),$$

$$\sum_{i=0}^n \nu(r, \mathbf{e}_i) x_i \geq \nu(r, n) \frac{\theta(r, n)}{\theta(0, n)} (x_0 + \dots + x_n) \quad (t < r \leq n)$$

Comparison of the LP-bounds

- ▶ We can show: Delsarte's LP bound implies our LP bound.
 - ▶ Hence: Delsarte's LP bound is potentially stronger.
 - ▶ To be investigated:
Concrete examples where the bounds really differ?
- ▶ But: Our bound works even when
 - ▶ there is no association scheme on Ω , or
 - ▶ the eigenvalues of the association scheme are unknown or too complicated.

Outline

Introduction

Designs in regular semilattices

A new 3 -(42 , 6 , 1) design

A new 3-(42, 6, 1) design

- ▶ Joint work with
Vedran Krčadinac and Alfred Wassermann.
- ▶ Small **open case** for Steiner systems:

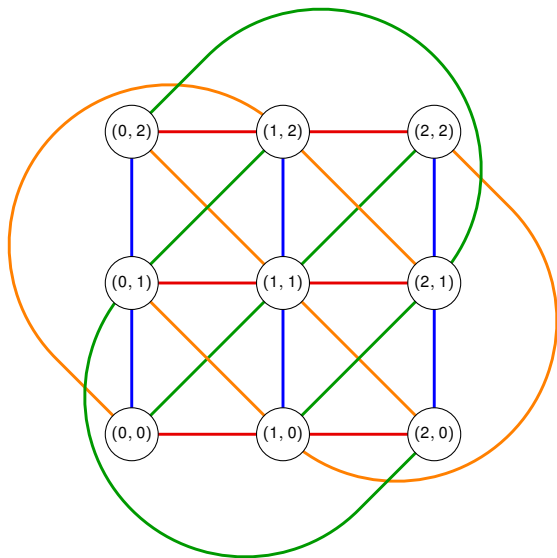
$$3-(42, 6, 1)$$

- ▶ K., Krčadinac, Wassermann 2025 by computer:
It exists!
- ▶ It is a **perfect** (42, 574, 8; 6) **constant-weight code**,
i. e. constant weight 6, length 42, mindist 8, size 574.
- ▶ Now: **Analyze** computer result.
- ▶ Potential reward:
 - ▶ structural insight,
 - ▶ **better**: computer-free construction,
 - ▶ **even better**: Infinite series of constructions.

Analysis

- ▶ Automorphism group G has order 432, isomorphic to affine linear group $AGL(2, 3)$.
- ▶ Suggests description based on $AG(2, 3)$ (affine plane of order 3, Hesse configuration).
- ▶ $AG(2, 3)$ has
 - ▶ 9 points,
 - ▶ 12 lines.

affine plane $AG(2, 3)$ aka Hesse configuration



- ▶ G has two orbits on V :
 - ▶ orbit V_A : size 18,
 - ▶ orbit V_B : size 24.
- ▶ Hence: from given group action,
 - ▶ identify the transitive action of $AGL(2, 3)$ on 18 elements,
 - ▶ identify the transitive action of $AGL(2, 3)$ on 24 elements,
 - ▶ understand the link of those actions.

Orbit V_A

- ▶ $18 = 2 \cdot 9$ suggests **signed points**.
- ▶ $\rightsquigarrow V_A = \mathbb{F}_3^\times \times \mathbb{F}_3 \times \mathbb{F}_3$.
- ▶ For $\mathbf{x} = (x_0, x_1, x_2) \in V_A$:
 - ▶ **sign** $\epsilon = x_0 \in \{\pm 1\}$,
 - ▶ **point** $P = \epsilon \cdot (x_1, x_2) \in \mathbb{F}_3^2$
 - ▶ notation P^ϵ .
- ▶ Realize $G \cong \text{AGL}(2, 3)$ as set of all matrices

$$\det(A) \cdot \begin{pmatrix} 1 & 0 \\ \mathbf{b} & A \end{pmatrix} \in \text{SL}(3, 3)$$

with $A \in \text{GL}(2, 3)$ and $\mathbf{b} \in \mathbb{F}_3^2$.

- ▶ Correct action of G on V_A : left multiplication.

Orbit V_B

- ▶ $24 = 2 \cdot 12 \rightsquigarrow$ signed lines?
Fails.
- ▶ $24 = 4 \cdot 3! \rightsquigarrow$ ordered parallel classes of lines?
Fails.
- ▶ Frustration.
- ▶ A few months later: parallel classes of triangles?
Works!!

Triangle classes

- ▶ Triangles Δ_1, Δ_2 **parallel**
 \iff their **sides** are **pairwise parallel**.
- ▶ Parallelism is equivalence relation
 \rightsquigarrow **parallel classes** of triangles (**triangle classes**).
- ▶ Special feature of \mathbb{F}_3 :
Each triangle class has size 3
and partitions the point set.

▶ Example:
$$\begin{pmatrix} A & B & A \\ B & B & C \\ A & C & C \end{pmatrix}$$

- ▶ Number of triangles $= \frac{9 \cdot 8 \cdot 6}{3!} = 72$.
 \implies number of triangle classes $= \frac{72}{3} = 24 \quad \checkmark$

Action on V_B

- ▶ Model affine plane $AG(2, 3)$ as V_A modulo sign.
(\rightsquigarrow standard embeddig of $AG(2, 3)$ in $PG(2, 3)$.)
- ▶ Let $V_B =$ set of **triangle classes** in $AG(2, 3)$.
- ▶ Induced action of G on V_B is correct!
- ▶ The actions on V_A and V_B are linked correctly!

Conclusion

We have identified the geometric ambient setting!

Reverse-engineering the computer result

- ▶ Blocks are 6-subsets of $V = V_A \cup V_B$.
- ▶ The 574 blocks split into 7 orbits.
- ▶ Goal: Geometric descriptions of orbit representatives.
- ▶ four “easy” orbits.
- ▶ three “harder” orbits, depend on orientation:
SL(2, 3)-action splits set of triangles
into “clockwise” and “counter-clockwise” ones.

Conclusion

- ▶ Geometric construction of a 3-(42, 6, 1) Steiner system.
- ▶ Natural $\text{AGL}(2, 3)$ -invariance.
- ▶ Proof of Steiner system property:
 - ▶ Show: all orbits of G on $\binom{V}{3}$ are covered.
 - ▶ “Exactly once” then follows from $\#D = 574$.
 - ▶ In principle possible by hand, by tedious analysis of 49 orbit representatives.
- ▶ Construction remains a miracle.
- ▶ Relies on sporadic properties of $\text{AG}(2, 3)$,
↪ no extension to an infinite series in sight.
- ▶ Remark. Orientation-flip yields a second, isomorphic 3-(42, 6, 1) design invariant under the given $\text{AGL}(2, 3)$ -action.
Computer search: these are the only two.

Thank you!

Slides will be uploaded at

<https://mathe2.uni-bayreuth.de/michaelk/>