On α-points of *q*-analogs of the Fano plane

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Subset lattice

- Let V be a v-element set.
- $\binom{V}{k} :=$ Set of all k-subsets of V.
- $\#\binom{V}{k} = \binom{v}{k}$.
- Subsets of V form a distributive lattice (wrt. $\subseteq$).

Definition

$D \subseteq \binom{V}{k}$ is a t-(v, k, λ) (block) design if
each $T \in \binom{V}{t}$ is contained in exactly λ blocks (elements of D).

Idea of q-analogs in combinatorics

Replace subset lattice by subspace lattice!

Subset lattice (repeated)

- Let V be a v-element set.
- $\binom{V}{k} :=$ Set of all k-subsets of V.
- $\#\binom{V}{k} = \binom{v}{k}$.
- Subsets of V form a distributive lattice (wrt. $\subseteq$).

Subspace lattice

- Let V be a v-dimensional $\mathbb{F}_q$ vector space.
- Grassmannian $\begin{bmatrix} V \\ k \end{bmatrix}_q :=$ Set of all k-dim. subspaces of V.
- Gaussian binomial coefficient

$$\#\begin{bmatrix} V \\ k \end{bmatrix}_q = \begin{bmatrix} v \\ k \end{bmatrix}_q = \frac{(q^v - 1)(q^{v-1} - 1) \cdot \ldots \cdot (q^{v-k+1} - 1)}{(q - 1)(q^2 - 1) \cdot \ldots \cdot (q^k - 1)}$$

- Subspaces of V form a modular lattice (wrt. $\subseteq$).

Projective geometry

- Subspace lattice of V = projective geometry $\mathrm{PG}(v-1,q)$
  - Elements of $\begin{bmatrix} V \\ 1 \end{bmatrix}_q$ are points.
  - Elements of $\begin{bmatrix} V \\ 2 \end{bmatrix}_q$ are lines.
  - Elements of $\begin{bmatrix} V \\ 3 \end{bmatrix}_q$ are planes.
  - Elements of $\begin{bmatrix} V \\ 4 \end{bmatrix}_q$ are solids.
  - Elements of $\begin{bmatrix} V \\ v-1 \end{bmatrix}_q$ are hyperplanes.

Advantage of geometric point of view

- Access to deep results
  developed in decades of research on finite geometries.
- Geometry provides *intuition*.

Attention!

- Dimensions are off by 1:

  Vector space of algebraic dimension v

  $\longleftrightarrow$ projective geometry of geometric dimension $v-1$.

Definition (block design, stated again)

Let V be a v-element set.
$D \subseteq \binom{V}{k}$ is a t-(v, k, λ) (block) design
if each $T \in \binom{V}{t}$ is contained in exactly λ elements of D.

q-analog of a design?

Definition (subspace design)

Let V be a v-dimensional $\mathbb{F}_q$ vector space.
$D \subseteq \begin{bmatrix} V \\ k \end{bmatrix}_q$ is a t-$(v, k, \lambda)_q$ (subspace) design
if each $T \in \begin{bmatrix} V \\ t \end{bmatrix}_q$ is contained in exactly λ elements of D.

- If $\lambda = 1$: D q-Steiner system
- If $\lambda = 1$, $t = 2$, $k = 3$: D q-Steiner triple system $\mathrm{STS}_q(v)$
- Geometrically:
  $\mathrm{STS}_q(v)$ is a set of planes in $\mathrm{PG}(v - 1, q)$
  covering each line exactly once.

Lemma

Let D be a t-$(v, k, \lambda)_q$ design and $i, j \in \{0, \ldots, t\}$ with $i + j \leq t$. Then for all $I \in \begin{bmatrix} V \\ i \end{bmatrix}_q$ and $J \in \begin{bmatrix} V \\ v-j \end{bmatrix}_q$ with $I \subseteq J$

$$\lambda_{i,j} := \#\{B \in D \mid I \subseteq B \subseteq J\} = \lambda \frac{\begin{bmatrix} v-i-j \\ k-i \end{bmatrix}_q}{\begin{bmatrix} v-t \\ k-t \end{bmatrix}_q}.$$

In particular, $\#D = \lambda_{0,0}$.

Corollary: Integrality conditions

If a t-$(v, k, \lambda)_q$ design exists, then all $\lambda_{i,j} \in \mathbb{Z}$.

Sufficient to check: $\lambda_i := \lambda_{i,0} \in \mathbb{Z}$ (Parameters admissible)

Corollary

$\mathrm{STS}_q(v)$ admissible $\iff$ $v \equiv 1, 3 \pmod 6$.

$\mathrm{STS}_q(v)$ for small admissible v

- $v = 3$
  $\mathrm{STS}_q(3) = \{V\}$ exists trivially.
- $v = 7$
  q-analog of the Fano plane $\mathrm{STS}_q(7)$.
  Existence undecided for every field order q.

  Most important open problem in q-analogs of designs.
- $v = 9$
  existence open for every q.
- $v = 13$
  $\mathrm{STS}_2(13)$ exists (Braun, Etzion, Östergård, Vardy, Wassermann 2013)
  Only known non-trivial STS_q.

q-Pascal triangle for $\mathrm{STS}_q(7)$ D

$$\lambda_{0,0} = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1$$

$$\lambda_{1,0} = q^4 + q^2 + 1 \qquad \lambda_{0,1} = q^5 + q^3 + q^2 + 1$$

$$\lambda_{2,0} = 1 \qquad \lambda_{1,1} = q^2 + 1 \qquad \lambda_{0,2} = q^2 + 1$$

- Each point P is contained in $\lambda_{1,0} = q^4 + q^2 + 1$ blocks.
- $\rightsquigarrow$ derived design wrt P ("local point of view from P")

$$\mathrm{Der}_P(D) = \{B/P \mid B \in D \text{ with } P \subseteq B\} \subseteq V/P$$

- In general: $\mathrm{Der}_P(D)$ is $(t-1)$-$(v-1, k-1, \lambda)_q$ design.
- $\implies$ $\mathrm{Der}_P(\mathrm{STS}_q(7))$ is 1-$(6, 2, 1)_q$ design.
  $=$ set of lines in $\mathrm{PG}(5, q)$ covering each point exactly once.
- In other words: $\mathrm{Der}(\mathrm{STS}_q(7))$ is a line spread of $\mathrm{PG}(5, q)$.

α-points

- spread $\mathcal{S}$ called geometric if for all distinct $L_1, L_2 \in \mathcal{S}$: $\{L \in \mathcal{S} \mid L \subseteq L_1 + L_2\}$ is spread of the solid $L_1 + L_2$.
- P is called α-point of $\mathrm{STS}_q(7)$ if the derived design in P is a geometric spread.
- S. Thomas 1996: There exists a non-α-point.
- O. Heden, P. Sissokho 2016: For $q = 2$: Each hyperplane contains non-α-point.
- Goal: Investigate Heden-Sissokho result for general q!

- Assume that H is hyperplane containing only α-points.
- Fix a poor solid S in H (not containing any block).
- Let $\mathcal{F} = \{F \in \begin{bmatrix} H \\ 5 \end{bmatrix}_q \mid S \subseteq F\}$.
  We have $\#\mathcal{F} = q + 1$.
- For $F \in \mathcal{F}$, let

$$\mathcal{L}_F := \{B \cap S \mid B \in D \text{ and } B + S = F\}.$$

- Lemma
  - $\mathcal{L}_F$ is a line spread of S.
  - The sets $\mathcal{L}_F$ with $F \in \mathcal{F}$ are pairwise disjoint.

Conclusion

$\mathcal{L} := \biguplus_{F \in \mathcal{F}} \mathcal{L}_F$ is a set of $(q+1)(q^2+1)$ lines in $\mathrm{PG}(3,q)$ admitting a partition into $q+1$ line spreads.

Lemma

For each point P in S, the $q+1$ lines in $\mathcal{L}$ passing through P span only a plane E_P.
(In other words, the lines form a pencil in E_P through P.)

Lemma

$(\begin{bmatrix} S \\ 1 \end{bmatrix}_q, \mathcal{L})$ is a projective generalized quadrangle of order (q,q).

Classification

Classification of projective generalized quadrangles:
(F. Buekenhout, C. Lefèvre 1974)
$\implies \left(\begin{bmatrix} S \\ 1 \end{bmatrix}_q, \mathcal{L}\right)$ is symplectic generalized quadrangle $W(q)$.

Implication

- By property of $\mathcal{L}$:
  The lines of $W(q)$ admit a partition into $q+1$ line spreads.
- Equivalently: The points of the parabolic quadric $Q(4, q)$ admit a partition into ovoids.
- Not possible for even q.
  - Payne, Thas: Finite generalized quadrangles, 3.4.1(i)
- Not possible for prime q.
  - Ball, Govaerts, Storme 2006:
    Each ovoid in $Q(4, q)$ is an elliptic quadric.
  - Any two of them have non-trivial intersection.

Theorem

Let q be prime or even and D a $\mathsf{STS}_q(7)$.
Then each hyperplane contains a non-α-point of D.

Research problem

- Investigate the remaining q
  (i.e. q a proper odd prime power).
- Can "each 5-subspace contains non-α-point" be shown?

arXiv preprint

`https://arxiv.org/abs/2105.00365`

Thank you!

Slides can be found at
`https://www.mathe2.uni-bayreuth.de/michaelk/`