On the lengths of divisible codes

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Linear codes

- finite field $\mathbb{F}_q$ of characteristic p.
- $\mathbb{F}_q$-linear code C: $\mathbb{F}_q$-subspace of $\mathbb{F}_q^n$.
- n: length of C.
- (Hamming) weight $w(\mathbf{c})$ of $\mathbf{c} \in \mathbb{F}_q^n$:
  $\#$ non-zero positions of $\mathbf{c}$.

Divisible codes

- Introduced by Harold Ward in 1981.
- Linear code C Δ-divisible $:\iff \Delta \mid w(\mathbf{c})$ for all $\mathbf{c} \in C$.
- Only interesting case: Δ power of p.
- In this talk: $\Delta = q^r \qquad (r \in \mathbb{N}_0)$.

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Why divisible codes?

- Many good codes are divisible.
- Connection to duality:
  Binary type II self-dual codes are 4-divisible.
  4-divisible binary codes are self-orthogonal.
  Self-orthogonal binary codes are 2-divisible.
  Self-orthogonal ternary codes are 3-divisible.
- Conjecture (Ward 2001):

  $$C \text{ Griesmer code over } \mathbb{F}_q, \quad p^r \mid \text{minimum distance of } C \implies C\ p^{r+1}/q\text{-divisible.}$$

  True for $q = p$ (Ward 1998), $q = 4$ (Ward 2001)
- Applications in finite geometry, subspace codes, etc.
  In this talk: Upper bounds for partial spreads.

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- Divisible code bound (Ward 1992):
  Bound on the dimensions of divisible codes.
- Our Goal:
  Classification of the effective lengths of q^r-divisible codes.

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Projective geometry

- $\mathbb{F}_q$-vector space V of dimension v.
- Subspace lattice of V: projective geometry $\mathrm{PG}(V)$
- 1-subspaces: points, $(v-1)$-subspaces: hyperplanes
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$$\begin{bmatrix} v \\ k \end{bmatrix}_q := \#(k\text{-subspaces of } V) = \begin{cases} \frac{(q^v-1)(q^{v-1}-1)\cdots(q^{v-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} & \text{if } 0 \leq k \leq v; \\ 0 & \text{otherwise.} \end{cases}$$

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Linear codes and points

- $\mathbb{F}_q$-linear code C of effective length n
  $\longleftrightarrow$ multiset $\mathcal{P}$ of n points in $\mathrm{PG}(V)$.
  (read columns of generator matrix
  as homogeneous coordinates)
- codeword $\mathbf{c}$ of C
  $\longleftrightarrow$ hyperplane H in $\mathrm{PG}(V)$
- $w(\mathbf{c}) = n - \#(\mathcal{P} \cap H)$.
- C Δ-divisible
  $\iff \#(\mathcal{P} \cap H) \equiv \#\mathcal{P} \pmod{\Delta}$ for all hyperplanes H.
  In this case: Call $\mathcal{P}$ Δ-divisible.
- $\rightsquigarrow$ Classify the sizes of q^r-divisible multisets of points!
  (will be called realizable sizes)

Advantages of geometric setting

- Basis-free approach to coding theory.
- Geometry provides *intuition*.

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Lemma

Let $V_1 \subseteq V_2$ $\mathbb{F}_q$-vector spaces and $\mathcal{P}$ multiset of points in V_1. Then:

$$\mathcal{P}\ q^r\text{-divisible in } V_1 \iff \mathcal{P}\ q^r\text{-divisible in } V_2$$

Lemma

Let U be $\mathbb{F}_q$-vector space of dimension $k \geq 1$.
Let $\mathcal{P}$ be the set of points in U.
Then $\mathcal{P}$ is q^{k-1}-divisible.

Proof.

Choose ambient space $V = U$. For each hyperplane H

$$\#(\mathcal{P} \cap H) = \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q = 1 + q + q^2 + \ldots + q^{k-2}$$

$$\equiv (1+q+q^2+\ldots+q^{k-2})+q^{k-1} = \begin{bmatrix} k \\ 1 \end{bmatrix}_q = \#\mathcal{P} \pmod{q^{k-1}}$$

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Lemma

The following sizes are realizable:

$$s(r,i) := q^i \cdot \begin{bmatrix} r-i+1 \\ 1 \end{bmatrix}_q = q^i + q^{i+1} + \ldots + q^r \quad (i \in \{0,\ldots,r\})$$

Proof.

Set of points of a $(r-i+1)$-subspace

is q^{r-i}-divisible of size $\begin{bmatrix} r-i+1 \\ 1 \end{bmatrix}_q$.

$\implies$ q^i-fold repetition

is $(q^i \cdot q^{r-i})$-divisible of size $q^i \cdot \begin{bmatrix} r-i+1 \\ 1 \end{bmatrix}_q$. □

Lemma

The following sizes are realizable:

$$n = a_0 s(r,0) + a_1 s(r,1) + \ldots + a_r s(r,r) \quad (a_0, a_1, \ldots, a_r \in \mathbb{N}_0)$$

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- The numbers

$$s(r,i) = q^i \cdot \begin{bmatrix} r-i+1 \\ 1 \end{bmatrix}_q = q^i + q^{i+1} + \ldots + q^r \quad (i \in \{0,\ldots,r\})$$

have the property

$$q^i \mid s(r,i) \qquad \text{but} \qquad q^{i+1} \nmid s(r,i).$$

- $\implies$ We can build positional number system upon base numbers

$$S(r) = (s(r,0), s(r,1), \ldots s(r,r))$$

- Each $n \in \mathbb{Z}$ has unique $S(r)$-adic expansion

$$n = a_0 s(r,0) + a_1 s(r,1) + \ldots + a_r s(r,r) \qquad (*)$$

with $a_0, \ldots, a_{r-1} \in \{0,\ldots,q-1\}$
and leading coefficient $a_r \in \mathbb{Z}$.
(Reason: Equation $(*)$ mod $q, q^2, q^3 \ldots$ yields unique $a_0, a_1, a_2, \ldots$)

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Example

- Let $q = 3$, $r = 3$. $\Longrightarrow$ $S(3) = (40, 39, 36, 27)$.
- $S(3)$-adic expansion of $n = 137$?
  Find $a_0, a_1, a_2 \in \{0, 1, 2\}$ and $a_3 \in \mathbb{Z}$ with

$$a_0 \cdot 40 + a_1 \cdot 39 + a_2 \cdot 36 + a_3 \cdot 27 = 137. \qquad (*)$$

- Modulo 3:

$$a_0 \cdot 1 + \underbrace{a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 0}_{=0} \equiv 2 \pmod 3 \quad \Longrightarrow \quad a_0 = 2$$

- $a_0 = 2$ in $(*)$:

$$a_1 \cdot 39 + a_2 \cdot 36 + a_3 \cdot 27 = \underbrace{137 - 2 \cdot 40}_{=57} \qquad (**)$$

- Modulo 9:

$$a_1 \cdot 3 + a_2 \cdot 0 + a_3 \cdot 0 \equiv 3 \pmod 9 \quad \Longrightarrow \quad a_1 = 1$$

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Example (cont.)

- … Find $a_1, a_2 \in \{0,1,2\}$ and $a_3 \in \mathbb{Z}$ with

$$a_1 \cdot 39 + a_2 \cdot 36 + a_3 \cdot 27 = 57. \qquad (**)$$

- $a_1 = 1$ in $(**)$:

$$a_2 \cdot 36 + a_3 \cdot 27 = \underbrace{57 - 1 \cdot 39}_{=18} \qquad (***)$$

- Modulo 27:

$$a_2 \cdot 9 + a_3 \cdot 0 \equiv 18 \pmod{27} \implies a_2 = 2$$

- In $(***)$:

$$a_3 \cdot 27 = \underbrace{18 - 2 \cdot 36}_{=-54} \implies a_3 = -2$$

- $\implies$ $S(3)$-adic expansion of $n = 137$ is

$$137 = 2 \cdot 40 + 1 \cdot 39 + 2 \cdot 36 + (-2) \cdot 27$$

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$$137 = 2 \cdot 40 + 1 \cdot 39 + 2 \cdot 36 + (-2) \cdot 27$$

Example (cont.)

- … Find $a_1, a_2 \in \{0, 1, 2\}$ and $a_3 \in \mathbb{Z}$ with

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Theorem 1

Let $n \in \mathbb{Z}$ and $r \in \mathbb{N}_0$. Then:

There exists a q^r-divisible $\mathbb{F}_q$-linear code of effective length n

$$\iff$$

The leading coefficient of the $S(r)$-adic expansion of n is ≥ 0.

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- $q = 3$, $r = 3$
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Lemma

Let $\mathcal{P}$ be non-empty and q^r-divisible.
Then for all hyperplanes H, $\mathcal{P} \cap H$ is q^{r-1}-divisible.

Proof of Theorem 1 (Idea)

- Let $\mathcal{P}$ be non-empty and q^r-divisible.
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Definition

- Let V be $\mathbb{F}_q$ vector space of dimension v.
- Let $\mathcal{S}$ be a set of k-subspaces of V.
- $\mathcal{S}$ is partial $(k-1)$-spread
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Năstase and Sissokho as a corollary from Theorem 1

- Let $\mathcal{S}$ be partial $(k-1)$-spread.
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$$S(k-1)\text{-adic ex.} = \sum_{i=0}^{k-2} (q-1)s(k-1,i) + \left(q \cdot \left(\begin{bmatrix} r \\ 1 \end{bmatrix}_q - k + 1 \right) - 1 \right) s(k-1,k-1)$$

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Projective divisible codes

- For partial spreads: $\mathcal{P}$ is a proper set (not only a multiset). Can we make use of this extra information?
- Sets of points $\longleftrightarrow$ projective linear codes.
- Classification of the lengths of projective q^r-divisible linear codes apparently much harder.

Theorem 2
There exists a projective 8-divisible binary linear code of length n

$$\iff n \notin \{1,2,3,4,5,6,7,\mathbf{8},9,10,11,\mathbf{12},13,\mathbf{14}\} \cup \{17,18,19,\mathbf{20},21,\mathbf{22},\mathbf{23},\mathbf{24},25,\mathbf{26},\mathbf{27},\mathbf{28},\mathbf{29}\} \cup \{33,\mathbf{34},\mathbf{35},\mathbf{36},\mathbf{37},\mathbf{38},\mathbf{39},\mathbf{40},\mathbf{41},\mathbf{42},\mathbf{43},\mathbf{44}\} \cup \{\mathbf{52},\mathbf{53},\mathbf{54},\mathbf{55},\mathbf{56},\mathbf{57},\mathbf{58},\mathbf{59}\}$$

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- Use first 4 MacWilliams-identities.
- Would be the size of the hole set of a partial 3-spread in $\mathbb{F}_2^{11}$ of size 133.
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- Hardest single case.
- Cannot have weights 56 and 48 (residuals would be proj. 4-divisible of length 3 and 11)
- If it has weight 40:
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