A computer approach to the enumeration of block designs which are invariant with respect to a prescribed permutation group

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Abstract

We present a new approach to the construction of simple block designs. Using the computer package DISCRETA, we start with the construction of block designs which are invariant with respect to some prescribed group of automorphisms. Therefore, one applies the method of Kramer and Mesner which means that one has to solve systems of diophantine equations to get the designs. DISCRETA has proven its usefulness for the construction of t-designs with large t and a prescribed group of automorphisms. In this paper, we generalize its application to the construction of designs on small point sets with no prescribed group. It is interesting to see that – to some extent – one can apply DISCRETA even in that extreme case.

But the solutions to the Kramer-Mesner system are not necessarily distinct from the point of view of the designs: Some designs may be obtained repeatedly and it is the question to find them and to put away duplicates. We present a new way for doing this by looking at the minimal subgroups of S_v, the symmetric group on all points. This approach avoids going through the whole lattice of subgroups which can be quite big even for symmetric groups of small degree. Using a combination of the computer packages GAP, GRAPE, nauty and COCO it is even possible to determine the full automorphism groups for all the isomorphism classes of designs. Small designs (with $v \leq 9$) are revisited and numerous new designs with the parameters $(11, 55, 20, 4, 6)$, $(11, 110, 40, 4, 12)$, $(13, 78, 30, 5, 10)$, $(14, 91, 26, 4, 6)$, $(14, 91, 39, 6, 15)$, $(15, 42, 14, 5, 4)$, $(15, 105, 35, 5, 10)$ and $(17, 68, 16, 4, 3)$ are discovered.

1 Introduction

Modern design theory has a long-standing history, the roots of which may be traced back to the XIX-th century. A classical treatise "Tactical memoranda" [Moo 896] by Eliakim Hastings Moore has certainly to be regarded as the main origin of the most part of notions, statements of the problems and even methods of construction in design theory (especially of Steiner systems). From the time of Moore one of the main tools for the construction of designs or, more generally, incidence structures (tactical configurations) was the use of convenient permutation groups. Conversely, the characterization of a prescribed permutation group (G, Ω) as the group of all automorphisms of a suitable incidence structure was always one of the most attractive ways to introduce (G, Ω). It was R.Carmichael who collected in his famous textbook [Car 37] a lot of impressive pairs: a design $\mathfrak{D}$ and its automorphism group $G = \mathrm{Aut}(\mathfrak{D})$, acting as a permutation group on the set Ω of the points of $\mathfrak{D}$.

Also from the time of Moore it was a desired task to enumerate designs with prescribed values of parameters up to isomorphism. In this paper we consider the problem of constructive enumeration of designs (in the sense of [Far 78a]). By that we mean the construction of a transversal of the set of all isomorphism classes of $2-(v, b, r, k, \lambda)$ designs with prescribed values of the parameters (here we restrict ourselves to the consideration of designs without repeated blocks only).

First attempts of constructive enumeration of designs with prescribed parameters were made many years ago, see e.g., [Nan 46a], [Nan 46b]. However only with the development of the computer era it became possible to attack this problem systematically. First serious achievements in this direction were reported in [Gib 76], [GibMC 77].

In 1978 I.A.Faradžev published (as the editor) a volume of papers [Far 78b], written by him, his students and coworkers. In this volume, in particular, the necessary background for a sufficiently effective computer constructive enumeration of graphs and incidence structures was presented. Roughly speaking, his approach consists of a backtracking algorithm for the construction of canonical (see [ArlZUF 74]) matrices (adjacency or incidence) which represent graphs or incidence structures respectively. Nowadays, such methodology is commonly regarded as orderly generation (the term goes back to [ColR 79]), see [Gol 92], [GruLM 97] for a brief account of the modern state of the art in the constructive enumeration of graphs. The enumeration of incidence structures is an essentially more difficult task than the enumeration of graphs (here we have "two degrees of freedom": labeling of vertices and blocks, while only one degree for graphs: labelling of vertices). In [Far 78b] only the main ideas were outlined (namely in the papers [IvaF 78], [ZaiF 78]). The crucial development of these ideas and implementations of the algorithms was done a few years later by A.V.Ivanov (see [Iva 80], [Iva 81]). Unfortunately, most part of these results appeared only in Russian (including a series of papers [IvaF 83], [Iva 83], publication of which was kindly initiated by H.-D.Gronau) and still are not known to western readers. Indeed, two rather short papers [Far 78a] and [Iva 85] show only the tip of a large iceberg. During a long time A.V.Ivanov was in a sense world champion in (exhaustive) constructive enumeration of small designs. Unfortunately, his computer programs were lost (perhaps forever) after the disappearing of the former USSR. In spite of the existence of some alternative program implementations (see [Gib 96]), this area of investigations still requires a modern advanced attack for which one of the sources may be a patient analysis of the achievements of the former Soviet school.

The orderly generation approach, as it was regarded above, allows (in principle) to obtain a complete list of all designs with prescribed parameters (if the computer will work as long as necessary). Practically, complete lists are available only for a rather small amount of parameter sets.

A well-known alternative approach may be identified, in a few words, as constructive enumeration of designs with prescribed parameters which are invariant with respect to a prescribed permutation group (H, Ω). This approach goes back to E.S.Kramer and D.M.Mesner ([KraM 76]), see Section 2.3 for details. In fact, the initial ideas of the method are rather simple, in different contexts they were discovered independently by a few authors, see e.g. [Kli 74]. A few attempts were also done to settle a computer implementation of the approach, one of the best programs was presented in [Kre 90].

Each program realization of the Kramer-Mesner method has to include two main routines: generation of all orbits of a group (H, Ω) on the set $\left\{ {\Omega \atop k} \right\}$ of all k-element subsets of Ω and a solver for a system of Diophantine equations (such a system gives a necessary and sufficient condition for the existence of a desired design). Traditional ways to solve both computational tasks were based on some combinatorial arguments, for example, change of base of a permutation group (see [BroFP 89]) was considered as a background for the generation of orbits on $\left\{ {\Omega \atop k} \right\}$.

A new approach toward the applications of computers in design theory was started a few years ago in Bayreuth. The first main innovation was based on the reduction of the problem of the determination of all $\{k\}$-orbits of (H, Ω) to the problem of enumeration of special double cosets in the symmetric group $S(\Omega)$ of the set Ω (see for details [Ker 91], [Sch 92]). For the latter problem there was elaborated a very effective technique which goes back to the paper [Lau 82] by R.Laue and which nowadays is called the "Leiterspiel" (snakes and ladders). A refinement and a first implementation of this new algorithmic approach was done by B.Schmalz, see [Sch 92], [Sch 93]. A new object oriented version is implemented by A. Betten.
The next crucial innovation was the creation of a few new solvers for the determination of all the solutions of the systems of Diophantine equations. The most powerful of them is (in general) the so-called LLL-solver, an implementation of the LLL-algorithm [LenLL 82], specially created by A.Wassermann for the use in the framework of Kramer-Mesner method (see [Was 97]).
The new computer package DISCRETA which was created in Bayreuth is in the process of permanent development. Its first version was described in [Lau 93], brief outline of the current status may be traced from [BetLW 97]. The main goal of the package was the discovering of new t-designs for t sufficiently large, see e.g. [Sch 93], [BetLW 96]. In our paper some other options for the use of this package will be considered.
The main goal of the users of DISCRETA is as a rule the proof of the existence of new designs, or more precisely of t-designs with new sets of parameters. The input of DISCRETA consists of a prescribed set of parameters and of a group (H, Ω). The output is the complete set of all t-designs with prescribed parameters which are invariant with respect to (H, Ω) (in general such a set may be empty). The points of all designs are labeled by elements of Ω. DISCRETA (in its current version) has no systematic tools for the testing of isomorphism of labeled designs. However, for a number of new t-designs with large t which were found by DISCRETA the use of group theoretic arguments allowed to solve the isomorphism problem without isomorphism testing. In most of such cases of a successful use of the package the group (H, Ω) was either a maximal subgroup of $S(\Omega)$, or the lattice of all overgroups of (H, Ω) in $S(\Omega)$ was known completely and was rather poor. The first case implies that all designs found by DISCRETA are pairwise non-isomorphic. In the second case some simple reasonings allow to count the number of non isomorphic designs (that is to solve the problem of the analytical enumeration of designs, see, e.g. [Sch 93]), while a constructive determination of representatives is

more elaborate. A deeper level of the use of group theoretic arguments for the solution of the isomorphism problem is presented in [BetKLW 97].
Generally, the problem of the analytical enumeration of orbits of an arbitrary group (H, Ω) may be solved by means of so-called Burnside marks (see [Ker 91], [FarKM 94]). If, in particular, (H, Ω) coincides with the identical subgroup of $S(\Omega)$, then one will count all designs up to isomorphism with prescribed parameters. For the goals of analytical enumeration of combinatorial objects the Burnside marks were firstly used (in explicit form) by M. Klin in [Kli 70] in the case of graphs. It turns out that in some extent a similar methodology was discovered in 1940 by the ingenious American mathematician J.H.Redfield (see [FarKM 94] for details). It was B.Schmalz who first used in [Sch 92] a similar methodology for the analytical enumeration of all designs with $v \leq 8$ (in particular, he used a table of marks for all 296 conjugacy classes of subgroups of S_8).
The constructive analogue of the above mentioned methodology was a few times discussed in literature in different contexts ([KliPR 88], [Ker 91], [Lau 93], [FarKM 94], etc.). A detailed and illustrated exposition of the main ideas may be found in [Lau 89]. Nevertheless, this methodology, to our knowledge, was never used practically for the purposes of the constructive enumeration of designs. We try to fill this gap in our paper. The reader will find a simple (perhaps even a bit naive) introduction to the methodology in Sections 2 and 3.
Sometimes we will face cases where the number of different labeled designs is counted by millions. In such cases the managing of all necessary data structures, testing of isomorphism, finding the automorphism group of a design are becoming very important stages of the job which require a lot of routine processing. We used for this purpose DISCRETA together with a few other computer packages, namely GAP, GRAPE, COCO, nauty. All these packages will be briefly described below.
The main purposes of this paper are

- to expose a methodology of the constructive enumeration of designs which are invariant with respect to a prescribed permutation group (H, Ω);
- to demonstrate practical application of this methodology by means of the existing computer packages - especially DISCRETA;
- to confirm known results (A.A.Ivanov, H.-D.Gronau and others) of the

constructive enumeration of small designs ($v \leq 9$), to supply these results by new information about the automorphism groups of all designs;

- to construct a reasonably large number of new designs for the sets of parameters for which before only a few (sometimes only one, perhaps with repeated blocks) designs were known;
- to formulate and to discuss some new interesting lines of development of DISCRETA, which are supposed to be created in the future, as a result of an analysis of all the advantages and disadvantages of our current computational abilities.

The sets of parameters, for which the results of constructive enumeration are reported below, may be classified into two parts. First part includes constructive enumeration of all designs, that is here (H, Ω) coincides with the identical subgroup. In these cases the use of orderly generation approach will be certainly much more effective; therefore in all such situations our intention was to test DISCRETA in "extreme circumstances" (in comparison with the traditional area of its applications). The second part includes a number of parameter sets with relatively small values, where the current knowledge of existing designs is rather poor. In all these cases we selected conveniently small groups (H, Ω) such that the use of DISCRETA turns out to be very effective. As a result the number of known designs is essentially increased.

The paper consists of 9 sections. In Section 2 all necessary preliminaries are briefly discussed. Section 3 is an exposition of a theoretical background of the used methodology; all computer facilities are discussed in Section 4. In Section 5 we consider just one rather striking example, namely how all 164 non-isomorphic $(8, 28, 14, 4, 6)$-designs (and their automorphism groups) were constructively enumerated, starting from the construction of 5,591,340 different labeled designs. In Section 6 all small designs are revisited, while in section 7 we enumerate designs for eight extra parameters with v points, $11 \leq v \leq 15$. Some historical notes and other remarks on the designs found are given in Section 8. Finally, in Section 9 we discuss with more details the past, present and future of a computer approach to the enumeration of designs. A few catalogues of designs being obtained are presented in a supplement.

On the whole, our exposition is based on a free use of elementary knowledge of group theory. As main sources books [Wie 64] and [Rot 95] may be

mentioned.
Our current project originates from a one-year teaching program which was arranged by M.Klin at Ben-Gurion University of the Negev, Beer-Sheva in 1995/96. The main participants of this program were a few German students, Ch.Pech one of them. Further visits of R.Laue to Beer-Sheva, Ch.Pech to Thurnau and Beer-Sheva, M.Klin to Dresden and Bayreuth have allowed to continue and finish the preparation of the paper.
It is worthwhile to mention that one of the origins of our interest to the enumeration of designs was the problem of description of designs by means of matrix algebras of different modes. Information about the designs obtained in current research gave some evidence for the prediction of "statistically regular" behavior of such algebras. Some results about so-called flag algebras of designs will be presented in [KliPZ].
We identify the style of this publication as an "experimental" paper (cf., e.g., the newly created journal "Experimental Mathematics"); by this reason we pay much attention (in particular in the final Section 9) to the discussion of methodology, comparison of different approaches to enumeration, careful description of all stages of computations, probable improvements of the used computer packages, etc.

2 Preliminaries

We start with the definitions and main notions used later on in this paper. In some cases further references are given, where a more detailed treatment of the matter can be found.

2.1 Permutation groups

Definition 1 *Let Ω be a set. A bijective mapping $\sigma : \Omega \longrightarrow \Omega$ is called a permutation of Ω, or a permutation (acting) on Ω.*

Lemma 2.1 *Let Ω be a set. Then the set $S(\Omega)$ of all permutations of Ω forms a group $(S(\Omega), \circ, {}^{-1})$. $\circ$ is the usual composition of mappings and ${}^{-1}$ denotes the inverse mapping.*

Remark: In all further considerations we will assume that Ω is a **finite** set. If in particular $\Omega = \{1, 2, \ldots, n\}$ then instead of $S(\Omega)$ we will also write S_n. Usually the sign $\circ$ for the composition of permutations is omitted.

Lemma 2.2 *A set $H \subseteq S(\Omega)$ forms a group if and only if it is closed with respect to the composition $\circ$. Such a group is called a* permutation group acting on Ω *and is usually denoted as (H, Ω).*

Definition 2 *(Group actions) Let G be a finite (abstract) group, Ω be a finite set. Let*

$$\varphi : G \times \Omega \longrightarrow \Omega$$

The triple (G, Ω, φ) is called an action of G on Ω if:

1) $\forall x \in \Omega \quad \varphi(e, x) = x$ (where e is the identical element of G),

2) $\forall x \in \Omega, \forall g, h \in G \quad \varphi(gh, x) = \varphi(h, \varphi(g, x))$.

Remarks:

- The result $\varphi(g, x)$ of the action of the group element g on $x \in \Omega$ is usually denoted as x^g.
- If φ is clear from the context we will write just (G, Ω) instead of (G, Ω, φ).

Definition 3 *An action φ is called faithful if*

$$(\forall x : \varphi(g, x) = x) \Longrightarrow g = e \quad .$$

Other notation and notions related to the use of permutation groups are described in [Wie 64], [KliPR 88], [Ker 91].
Concluding we introduce some standard actions that will be frequently needed in this paper.

Definition 4 *Let (G, X) and (H, Y) be two permutation groups. Let $m = |X|$ and $n = |Y|$. Without loss of generality, assume that $X \cap Y = \emptyset$. Following we define two different actions of the (abstract) direct product of G and H:*

$G + H := (G \times H, X \cup Y, \varphi_1)$ where

$$\varphi_1((g, h), x) = \begin{cases} x^g & \text{if } x \in X \\ x^h & \text{if } x \in Y \end{cases} \quad .$$

$G + H$ is called the direct sum *of the permutation groups (G, X) and (H, Y).*

$G \times H := (G \times H, X \times Y, \varphi_2)$ where

$$\varphi_2((g,h),(x,y)) = (x^g, y^h).$$

$G \times H$ is called the direct product of the permutation groups (G,X) and (H,Y).

Here and below let $\mathbb{Z}_n$ be the cyclic group on n points. We define Id_n to be the trivial group acting on an n-point set.

Example 2.1 The group $\mathbb{Z}_2 + \mathbb{Z}_2$ is $\langle(1\,2),(3\,4)\rangle$.
The group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is $\langle(1\,2)(3\,4),(1\,3)(2\,4)\rangle$.
The group $\mathbb{Z}_2 + Id_2$ is $\langle(1\,2)(3)(4)\rangle$.
The group $\mathbb{Z}_2 \times Id_2$ is $\langle(1\,2)(3\,4)\rangle$.

2.2 Designs

Here and below for a set A and for $k \in I\!N$ we denote by $\left\{ {A \atop k} \right\}$ the set of all k–element subsets of A.

Definition 5 *Let $\mathcal{P}$ be a finite set (of points), $v := |\mathcal{P}|$. Let $\mathcal{B}$ be a finite set (of blocks), $b := |\mathcal{B}|$. Let $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$ be an incidence relation.*
The triple $\mathfrak{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is called block design (or just design) if

1) any block $a \in \mathcal{B}$ is incident to the same number k of points (where k is a fixed number and $k < v$);

2) any point $A \in \mathcal{P}$ lies on the same number r of blocks (where r is a fixed number and $r < b$);

3) for any pair $\{A, B\} \in \left\{ {\mathcal{P} \atop 2} \right\}$ there exist λ blocks that contain A and B simultaneously.

The tuple (v, b, r, k, λ) is called the set of main parameters of $\mathfrak{D}$.

Remarks:

- It could happen that there exist some blocks which are incident with exactly the same points. In this case $\mathfrak{D}$ is called a design with repeated blocks.

- A common and useful specialization of the above definition is obtained by replacing condition 3 with:

  3′) for a fixed $t \geq 2$ and any t-subset of points $\{A_1, A_2, \ldots, A_t\}$ there exist exactly λ_t blocks that contain all of these points simultaneously.

  Such designs are called t-designs with parameters $t - (v, b, r, k, \lambda_t)$.

  Any t-design is also a t'-design for $t' \leq t$.

In what follows, we are interested in designs without repeated blocks only. In this case one can identify $\mathcal{B}$ with a set of k–element subsets of $\mathcal{P}$ and forget about $\mathcal{I}$ (the incidence is naturally given by the ϵ–relation, that is by the relation of inclusion).

Definition 6 *Let $\mathfrak{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, \mathcal{I}_1)$ and $\mathfrak{D}_2 = (\mathcal{P}_2, \mathcal{B}_2, \mathcal{I}_2)$ be designs.*
A bijective mapping σ that maps $\mathcal{P}_1$ to $\mathcal{P}_2$ and $\mathcal{B}_1$ to $\mathcal{B}_2$ is called isomorphism between $\mathfrak{D}_1$ and $\mathfrak{D}_2$ *if and only if*

$$\forall A \in \mathcal{P}_1 \quad \forall a \in \mathcal{B}_1 : (A, a) \in \mathcal{I}_1 \iff (A^\sigma, a^\sigma) \in \mathcal{I}_2 \quad .$$

In this situation we will later on use the notation $\mathfrak{D}_1^\sigma = \mathfrak{D}_2$ or $\mathfrak{D}_1 \cong \mathfrak{D}_2$.

Remark: From a general point of view, two isomorphic designs $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are indistinguishable by algebraical means. That is why in general they are identified. However, sometimes it turns out to be very important to distinguish also different but isomorphic designs. In such cases we will explicitly speak about labeled designs.

Definition 7 *An isomorphism that maps a design $\mathfrak{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ onto itself is called automorphism.*

Remark: In the case of designs without repeated blocks an isomorphism (and in particular each automorphism) is already uniquely defined by its action on the points (this action is faithful in this case).
At the end of this section we will give a simple structural lemma. It yields a connection between the parameters of a design and the relative constellation of its blocks to each other.

Lemma 2.3 *Let $\mathfrak{D}$ be a design with the parameters (v, b, r, k, λ). Let $b_0 \in \mathcal{B}$ and let*

$$n_i := |\{b \in (\mathcal{B} \setminus \{b_0\}) : |b \cap b_0| = i\}| \quad .$$

Then the following is true:

1) $\sum_{i=0}^{k} n_i = b - 1$;

2) $\sum_{i=0}^{k} i n_i = (r - 1)k$;

3) $\sum_{i=0}^{k} i(i - 1)n_i = (\lambda - 1)(k^2 - k)$.

□:
This can be proved by the use of some simple combinatorial arguments. Namely, counting in two different ways the number of blocks, flags and double flags of the given design $\mathfrak{D}$ gives the results.

:□

For a more detailed treatment of these and other questions related to designs we refer to [Hal 86], [HugP 85] and [BetJL 93].

2.3 Designs and permutation groups

2.3.1 k–orbits and $\{k\}$–orbits

Definition 8 *Let (G, Ω) be a permutation group. Let $x \in \Omega$.*
Then the set $\{x^g : g \in G\}$ is called an orbit of (G, Ω) (that is generated by x).

For a given permutation group G acting on a set Ω it is possible to define a few naturally induced actions on other objects: e.g. on k-tuples or on k–element subsets of Ω.

Lemma 2.4 *Let (G, Ω) be a permutation group. Let $0 < k \leq |\Omega|$. Then*

$$\begin{aligned} \varphi : G \times \Omega^k &\longrightarrow \Omega^k, \\ \varphi\left(g, (x_1, x_2, \ldots, x_k)\right) &:= ({x_1}^g, {x_2}^g, \ldots {x_k}^g) \end{aligned}$$

and

$$\begin{aligned}\tilde{\varphi} : G \times \left\{ \begin{matrix} \Omega \\ k \end{matrix} \right\} &\longrightarrow \left\{ \begin{matrix} \Omega \\ k \end{matrix} \right\}, \\ \tilde{\varphi}\left(g, \{x_1, x_2, \ldots, x_k\}\right) &:= \{{x_1}^g, {x_2}^g, \ldots, {x_k}^g\}\end{aligned}$$

yield actions of G on Ω^k or $\left\{ \begin{smallmatrix} \Omega \\ k \end{smallmatrix} \right\}$, respectively.

Definition 9 *Orbits of $\left(G, \Omega^k\right)$ are called k–orbits and orbits of $\left(G, \left\{ \begin{smallmatrix} \Omega \\ k \end{smallmatrix} \right\}\right)$ are called $\{k\}$–orbits. If $k = 1$ then we just use the term orbit instead of* 1*-orbit or* $\{1\}$*-orbit.*

For more details about invariant k–relations and the corresponding Galois correspondences see [Wie 69], [KalK 72] and [FarKM 94].

2.3.2 KM–matrices and the Kramer-Mesner method

Here we will give a short description of the Kramer-Mesner method for the construction of labeled designs which are invariant with respect to a given permutation group.
Roughly speaking it uses the following relatively simple observation:
If $\mathfrak{D} = (\mathcal{P}, \mathcal{B})$ is a t–design with parameters (v, b, r, k, λ) and $(H, \mathcal{P}) \leq \mathrm{Aut}(\mathfrak{D})$ then $\mathcal{B}$ can be expressed as the union of some $\{k\}$–orbits of $(H, \mathcal{P})$. The Kramer-Mesner method gives a way to find selections of $\{k\}$–orbits of $(H, \mathcal{P})$ that yield t–designs:

Lemma 2.5 *Let (H, Ω) be a permutation group and $0 < t < k \leq |\Omega|$. Let $\mathcal{O}_1$ be a $\{t\}$–orbit, $\mathcal{O}_2$ be a $\{k\}$–orbit of (H, Ω) and $a \in \mathcal{O}_1$. Let*

$$m_{\mathcal{O}_1,\mathcal{O}_2}(a) := |\{x \in \mathcal{O}_2 : a \subseteq x\}| \quad .$$

Then $m_{\mathcal{O}_1,\mathcal{O}_2}(a)$ does not depend on the particular choice of $a \in \mathcal{O}_1$. That is why it will be denoted as $m_{\mathcal{O}_1,\mathcal{O}_2}$.

Remark: From this follows that to each pair $(\mathcal{O}_1, \mathcal{O}_2)$ of a $\{t\}$–orbit and a $\{k\}$–orbit of (H, Ω) one can assign a number $m_{\mathcal{O}_1,\mathcal{O}_2}$ as was done above.

Definition 10 *Let* (H, Ω) *be a permutation group and* $0 < t < k \leq |\Omega|$. *Let* $\mathfrak{S}_{\{t\}}$, $\mathfrak{S}_{\{k\}}$ *be the systems of* $\{t\}$*– and* $\{k\}$*–orbits, respectively.*
The matrix

$$\mathfrak{K} = (m_{\mathcal{O}_1, \mathcal{O}_2})_{\mathcal{O}_1 \in \mathfrak{S}_{\{t\}}, \mathcal{O}_2 \in \mathfrak{S}_{\{k\}}}$$

is called KM-matrix related to (G, Ω) *and the given values of* t *and* k.

Remark: The ordering of rows and columns of $\mathfrak{K}$ reflects the ordering of orbits in $\mathfrak{S}_{\{t\}}$ and $\mathfrak{S}_{\{k\}}$. That is, $\mathfrak{K}$ is uniquely defined only up to permutations of rows and columns.

Finally the following theorem gives the connection between KM-matrices and the construction of designs:

Theorem 2.6 (E.S. Kramer and D.M. Mesner [KraM 76])
Let (H, Ω) *be a permutation group. For* $0 < t < k \leq |\Omega| = v$ *let* $\mathfrak{K} \in \mathbb{N}^{l \times c}$ *be the corresponding KM-matrix (here* $l = \left|\mathfrak{S}_{\{t\}}\right|$ *and* $c = \left|\mathfrak{S}_{\{k\}}\right|$*). Then the solutions of the following Diophantine system of equations give all different labeled* t*-designs with parameters* (v, b, r, k, λ_t) *which are invariant with respect to* (H, Ω)*:*

$$\mathfrak{K}x = \begin{pmatrix} \lambda_t \\ \lambda_t \\ \vdots \\ \lambda_t \end{pmatrix} \qquad \textit{where } x \in \{0,1\}^l$$

Here the fact $x_i = 1$ *in a solution means that the corresponding* k*-orbit of* (H, Ω) *is a part of* $\mathcal{B}$.

Remarks:

- Clearly the resulting designs are invariant with respect to H.
- Note that in general this method gives a lot of isomorphic solutions. This is why it is mainly used for the proof of the existence of designs with certain interesting parameters (e.g. the search for t-designs with large t but relatively small values of v and λ_t).
- The automorphism groups of the designs which are obtained may in general be larger than H.

Concluding this section we will give an illustrative example for the Kramer-Mesner method:

Example 2.2 Let

$$H = D_7 = \langle (1,2,3,4,5,6,7),(2,7)(3,6)(4,5)\rangle\,,$$

that is D_7 is the dihedral transitive permutation group of degree 7 and of order 14. We would like to construct 2-designs with $k = 3$, $\lambda = 2$ that are invariant with respect to H. For this we have to compute $\{2\}$-orbits and $\{3\}$-orbits of H. In what follows $\mathcal{O}^{(i)}_{\{l\}}$ denotes the i-th $\{l\}$-orbit of H (the order in which we write these orbits is in fact arbitrary but we have to fix it in the beginning):

$$\begin{aligned}
\mathcal{O}^{(1)}_{\{2\}} &= \{\{1,2\},\{1,7\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,7\}\}\,,\\
\mathcal{O}^{(2)}_{\{2\}} &= \{\{1,3\},\{1,6\},\{2,4\},\{2,7\},\{3,5\},\{4,6\},\{5,7\}\}\,,\\
\mathcal{O}^{(3)}_{\{2\}} &= \{\{1,4\},\{1,5\},\{2,5\},\{2,6\},\{3,6\},\{3,7\},\{4,7\}\}\,.
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}^{(1)}_{\{3\}} &= \{\{1,2,3\},\{1,2,7\},\{1,6,7\},\{2,3,4\},\{3,4,5\},\{4,5,6\},\{5,6,7\}\}\,,\\
\mathcal{O}^{(2)}_{\{3\}} &= \{\{1,2,4\},\{1,2,6\},\{1,3,4\},\{1,3,7\},\{1,5,6\},\{1,5,7\},\{2,3,5\},\\
&\quad\ \{2,3,7\},\{2,4,5\},\{2,6,7\},\{3,4,6\},\{3,5,6\},\{4,5,7\},\{4,6,7\}\}\,,\\
\mathcal{O}^{(3)}_{\{3\}} &= \{\{1,2,5\},\{1,4,5\},\{1,4,7\},\{2,3,6\},\{2,5,6\},\{3,4,7\},\{3,6,7\}\}\,,\\
\mathcal{O}^{(4)}_{\{3\}} &= \{\{1,3,5\},\{1,3,6\},\{1,4,6\},\{2,4,6\},\{2,4,7\},\{2,5,7\},\{3,5,7\}\}\,.
\end{aligned}$$

Now, to get the KM-matrix we have to count the number of appearances of representatives from the $\{2\}$-orbits in the $\{3\}$-orbits. For instance $\{1,2\}$ from $\mathcal{O}^{(1)}_{\{2\}}$ appears 2 times in $\mathcal{O}^{(1)}_{\{3\}}$, namely in $\{1,2,3\}$ and in $\{1,2,7\}$. Thus the first entry in the first row of $\mathfrak{K}$ has to be 2. We get

$$\begin{array}{c|cccc}
 & \mathcal{O}^{(1)}_{\{3\}} & \mathcal{O}^{(2)}_{\{3\}} & \mathcal{O}^{(3)}_{\{3\}} & \mathcal{O}^{(1)}_{\{3\}} \\
\mathcal{O}^{(1)}_{\{2\}} & 2 & 2 & 1 & 0 \\
\mathcal{O}^{(2)}_{\{2\}} & 1 & 2 & 0 & 2 \\
\mathcal{O}^{(3)}_{\{2\}} & 0 & 2 & 2 & 1
\end{array}\,.$$

Thus

$$\mathfrak{K} = \begin{pmatrix} 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}.$$

Now we have to search for 0-1-solutions of the following system of equations:

$$\mathfrak{K}x = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

Clearly the only solution is $x = (0,1,0,0)^t$. This means that $\mathcal{O}^{(2)}_{\{3\}}$ forms the block set of a design with the desired properties.

3 Theoretical background

Suppose we are given a set of parameters (v, b, r, k, λ) and a permutation group $H \leq S_v$. Moreover, suppose we already know the complete list $\mathcal{L}$ of labeled designs with these parameters being invariant with respect to H.
Our goal is to decompose $\mathcal{L}$ into isomorphism classes. The most straightforward way to achieve this would be to make a complete exhaustive search for non isomorphic designs using isomorphism test (as it is provided by nauty; see section 4). While this may sound good for a theoretical approach it is usually inappropriate for an algorithmic approach, since $\mathcal{L}$ may consist of say several millions of designs. Therefore ways have to be found to reduce the number of objects to be "honestly" checked against isomorphism as far as possible.
A good way to achieve this is to "pre-partition" the given set $\mathcal{L}$ of designs into sufficiently large classes of isomorphic designs. The resulting (intermediate with respect to our goal) partition π of $\mathcal{L}$ may in general be finer than the desired partition of $\mathcal{L}$ into isomorphism classes. Then to solve the isomorphism problem in $\mathcal{L}$ one has to compare only the representatives of a transversal of the partition π.
The following simple propositions may help to get a good start for the use of the above approach.

Lemma 3.1 *Let $\mathfrak{D}$ be a design, $\varphi \in S_v$, $\mathfrak{D}' = \mathfrak{D}^\varphi$ and $G = \mathrm{Aut}(\mathfrak{D})$.*
Then

$$\mathrm{Aut}(\mathfrak{D}') = \varphi^{-1} G \varphi\,.$$

Lemma 3.2 *Let $\mathfrak{D}$ be a design with parameters (v, b, r, k, λ) which is invariant with respect to a group $H \leq S_v$.*
Let $N := \mathfrak{N}_{S_v}(H)$ be the normalizer of H in S_v and $\varphi \in N$.
Then $\mathfrak{D}^\varphi$ is also invariant with respect to H.

□:

$$H \le \mathrm{Aut}(\mathfrak{D}) \Longrightarrow \varphi^{-1} H \varphi \le \mathrm{Aut}(\mathfrak{D}^{\varphi}) \Longrightarrow H \le \mathrm{Aut}(\mathfrak{D}^{\varphi})\,.$$

:□

Corollary 3.3 *Let $H \le S_v$ and $\mathcal{L}$ be the class of all labeled designs (with given parameters) being invariant with respect to H. Let $N = \mathfrak{N}_{S_v}(H)$.*
Let $O_1, O_2, \ldots, O_n$ be all the orbits of the action of N on $\mathcal{L}$.
Then

$$\textit{For } 1 \le k \le n : \mathfrak{D}_1, \mathfrak{D}_2 \in O_k \Longrightarrow \mathfrak{D}_1 \cong \mathfrak{D}_2 \quad .$$

Remark: The opposite implication does not hold in general. This means there may be designs that are in different orbits of N but that are still isomorphic. However, that happens only for designs whose automorphism group is strictly larger than H, see Corollary 3.7 below.

If the size of $\mathcal{L}$ is *very* large it may happen that the computation of orbits under N can not be handled even by todays comparatively large computers (because of lack of memory). In this case we have to split the task into several smaller parts to reduce the effort (divide et impera).
One idea is, to proceed separately with all designs whose automorphism group is a *proper* overgroup of H and with those having *exactly* H as automorphism group. So, an obvious strategy would be to determine evidently all overgroups of H and classify the designs fixed by H according to their automorphism group first. Then the orbits of N on the set of the designs with exact automorphism group H would be the desired isomorphism classes.
Here we choose a different strategy, in that only some small overgroups of H are chosen and all other overgroups are mot examined. In particular, no knowledge of the full lattice of subgroups of S_v is needed.
With such a goal we make the following simple observations.

Lemma 3.4 *Let $\mathfrak{D}$ be a design with parameters (v, k, λ) and $H < \mathrm{Aut}(\mathfrak{D})$. Then there exists a minimal proper overgroup $\mathcal{O}$ of H such that $\mathcal{O} \le \mathrm{Aut}(\mathfrak{D})$.*

□:
This is trivial since we are dealing with subgroups of finite groups.

:□

In fact, since a design $\mathfrak{D}$ with parameters (v, b, r, k, λ) (having no repeated blocks) can be expressed as a totally symmetric k-ary relation over a v-element set, the previous lemma can be formulated a bit stronger by exchanging "minimal overgroup" with "minimal k-closed overgroup" (see [Wie 69], [FarKM 94] for definitions).
Using Lemma 3.1 we can restrict ourselves to the consideration of representatives of conjugacy classes of minimal overgroups of G.

Lemma 3.5 *Let $H \leq S_v$, $\mathfrak{D}$ be a design with parameters (v, b, r, k, λ) which is invariant with respect to H. Let $\mathcal{M}$ be a transversal of the conjugacy classes of the proper minimal overgroups of H.*
Then either H is the full automorphism group of $\mathfrak{D}$ or there exist a $\varphi \in S_v$ and a $U \in \mathcal{M}$ such that

$$\mathrm{Aut}(\mathfrak{D}^\varphi) \geq U \quad .$$

□:
The proof is evident.

:□

Now we can formulate our strategy of the algorithm for the constructive enumeration of all designs $\mathfrak{D}$ with $\mathrm{Aut}(\mathfrak{D}) > H$. Namely:

- find a transversal $\mathcal{M}$ of proper minimal overgroups of H;
- for each $G \in \mathcal{M}$ compute a transversal $\mathcal{T}_G$ of isomorphism classes of designs being invariant with respect to G;
- Merge all $\mathcal{T}_G$ and do a complete search for isomorphism classes.

Especially the last step is easy to fulfill since we can assume that each $\mathcal{T}_G$ consists of canonically labeled designs (nauty provides them when doing isomorphism check).
The only thing that still has to be described is how to enumerate the representatives of designs having exactly H as automorphism group.
Suppose we have a list $\mathcal{L} = \mathcal{L}_H$ of all labeled designs being invariant with respect to H. With the methods mentioned before we can get a complete list of mutually non-isomorphic designs having a proper overgroup of H as automorphism group.
The following lemma allows to count the number of designs having exactly H as automorphism group.

Lemma 3.6 *Let $H \leq S_v$ and $\mathfrak{D}$ be a design with* $\mathrm{Aut}(\mathfrak{D}) = H$*. Then the size of the isomorphism class containing $\mathfrak{D}$ is*

$$[S_v : H].$$

In particular, $[\mathfrak{N}_{S_v}(H) : H]$ designs from this class have exactly H as the automorphism group.

□

Corollary 3.7 *Let $H \leq S_v$, $\mathcal{L}$ be the class of all labeled invariant designs of H. Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be the class of all labeled designs whose automorphism group is a proper overgroup of H.*
Then the number of isomorphism classes of designs having exactly automorphism group H is

$$\frac{|\mathcal{L}| - |\mathcal{L}_0|}{[\mathfrak{N}_{S_v}(H) : H]}$$

□

A similar way leads to the constructive enumeration of designs having exactly H as automorphism group. For this we take $\mathcal{L} \setminus \mathcal{L}_0 =: \mathcal{L}_1$ and compute the orbits of $(\mathfrak{N}_{S_v}(H), \mathcal{L}_1)$.

4 Computer facilities

In most cases, the construction of isomorphism types of designs is certainly not feasible by hand computations. Luckily, in the last few years a lot of computer packages for algebraic and combinatorial calculations were developed. While each of them alone is powerful in its own special area of applications, used in conjunction they become quite a force.
In the following we will give very brief outlines of a set of computer packages which are suitable for the realization of the methods described above.

- GAP (**G**roups, **A**lgorithms, **P**rograms)
  is a computer package for the computations with algebraic objects such as permutation groups, finitely represented groups, rings, fields etc. It was created at the RWTH Aachen (Germany) (see [Schö 95]).

  For our purposes its abilities to handle finite permutation groups become especially useful. E.g. the computation of normalizers of groups or of orbits of designs can be done very conveniently.

- nauty (**n**o **aut**omorphisms, **y**es?)
  is a program for the computation of automorphism groups of vertex-coloured graphs as well as for the isomorphism test of graphs. It was written by B. McKay (see [McK 90]).

  Since designs can be represented as graphs (to each design $\mathfrak{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ corresponds uniquely the total point graph $\Gamma_{\mathfrak{D}} = (\mathcal{P} \dot{\cup} \mathcal{B}, \mathcal{I} \dot{\cup} (\mathcal{P} \times \mathcal{P})))$, it can also be used for the isomorphism test of designs.

  nauty is integrated into GAP by means of the package GRAPE which in turn provides algorithms for the computations with graphs (it was written by L.H. Soicher [Soi 93]). However, for our purposes we only used the interface to nauty that is provided by GRAPE.

- COCO (**CO**herent **CO**nfigurations)
  The computer package COCO (written by I.A. Faradžev and M.H. Klin; Unix implementation by A.E. Brouwer) establishes the Galois correspondence between permutation groups and coherent (cellular) rings (see [FarKM 94], [FarK 91] and [FarIK 90]). It consists of 5 major parts:

  - Inducing of permutation groups on combinatorial objects,
  - Computation of the coloured graph corresponding to the centralizer ring of a permutation group,
  - Computation of the intersection numbers of a cellular ring,
  - Computation of the cellular subrings of cellular rings,
  - Computation of the automorphism group of cellular rings.

  In our case the most important part is the first one since it provides a very fast orbit algorithm (quite faster than the one available via GAP) for the construction of the orbit of a prescribed labeled design $\mathfrak{D}$ with v points under the symmetric group S_v. Also the established Galois correspondence may be useful when computing the minimal overgroups of a group.

- DISCRETA
  is (for our goals) surely the most important part of our tool box. Given a set of parameters (v, b, r, k, λ) and a permutation group $H \leq S_v$ it

enumerates, using the KM–method all labeled designs with the given parameters being invariant with respect to H.

It was written at the University of Bayreuth (Germany) (see [BetLW 96], [BetLW 97] for more details). One important feature of this system is its graphical user interface which allows to use it without writing programs.

Since DISCRETA is still under development, some desired facilities are still lacking. So it is e.g. difficult to interface it with other computer packages such as GAP. This can be circumvented by writing small specialized c-programs.

5 Constructive Enumeration of $(8, 28, 14, 4, 6)$-designs

In [Iva 80], [Iva 85] A.V. Ivanov presented his algorithm for the constructive enumeration up to isomorphism of designs with given parameters. Part of his work was the computer implementation of his algorithm on an IBM 370/165 (circa 3 MIPS).
With this program he undertook the enumeration of designs with small parameters. Part of his results are listed in Table 8.1.
The first set of parameters for which the constructive enumeration was not successful for some reason is $(8, 28, 14, 4, 6)$.
Later, in 1981, all these designs were constructively enumerated by H.-D.Gronau and R.Reimer in [GroR 81].
In 1992 B.Schmalz described an algorithm for the analytical and constructive enumeration of designs (up to isomorphism) which are invariant with respect to a prescribed permutation group. He undertook the analytical enumeration of the $(8, 28, 14, 4, 6)$-designs (amongst other parameter sets) and the number coincided with the one from [GroR 81].
We will use this parameter set to test and to illustrate the strategy that was described in Section 3.
The reason why we are doing this, is, that from one hand the parameter set $(8, 28, 14, 4, 6)$ is the smallest "interesting" set (here a lot of designs appear). From the other hand it seems to be a good idea to reconfirm the constructive results from [GroR 81].

The information we are starting with is the parameter set $(8, 28, 14, 4, 6)$ and the group $H = id < S_8$.
Using DISCRETA one finds the number of labeled designs with these parameters. It is 5,591,340. This huge number makes it impractical to proceed by brutal force and to compute just the orbits of those designs under S_8. The lattice of all subgroups of S_8 is rather big so it is difficult to investigate designs invariant with respect to each subgroup of S_8. In what follows, we start by looking only at the minimal subgroups, e.g. those which are minimal overgroups of id.
Minimal subgroups have prime order. Since $8! = 40320 = 2^7 \cdot 3^2 \cdot 5 \cdot 7$ the minimal subgroups are $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$ and $\mathbb{Z}_7$ as abstract groups. The following table lists a set of representatives H for the conjugacy classes of minimal sbgroups of S_8. Call the list of groups $\mathcal{M}$. In the second column, the isomorphism type as a permutation group of the normalizer is shown. For the proof, see, e.g. [KliPR 88], [Ker 91]. Warning: we use the notation for the wreath product as in [KliPR 88]).

representative H	isomorphism type of $\mathfrak{N}_{S_v}(H)$
$\langle(1,2)\rangle$	$\mathbb{Z}_2 + S_6$
$\langle(1,2)(3,4)\rangle$	$(\mathbb{Z}_2 \wr \mathbb{Z}_2) + S_4$
$\langle(1,2)(3,4)(5,6)\rangle$	$(S_3 \wr \mathbb{Z}_2) + \mathbb{Z}_2$
$\langle(1,2)(3,4)(5,6)(7,8)\rangle$	$(S_4 \wr \mathbb{Z}_2)$
$\langle(1,2,3)\rangle$	$\mathrm{AGL}(1,3) + S_5$
$\langle(1,2,3)(4,5,6)\rangle$	$(\mathbb{Z}_2 \wr \mathrm{AGL}(1,3)) + \mathbb{Z}_2$
$\langle(1,2,3,4,5)\rangle$	$\mathrm{AGL}(1,5) + S_3$
$\langle(1,2,3,4,5,6,7)\rangle$	$\mathrm{AGL}(1,7)$

Again using DISCRETA one finds for each group H of $\mathcal{M}$ the designs which are invariant under H. Denote the set of solutions for fixed group H by $\mathcal{L}_H$. For $H \in \mathcal{M}$, the numbers are reasonable and we can proceed determining isomorphism types using normalizers and the package nauty as it was described in Section 3. The results are listed in the following table.

Group H	$\lvert\mathcal{L}_H\rvert$	# of N_H orbits on $\mathcal{L}_H$	# of iso-classes
$\langle(1,2)\rangle$	2460	6	6
$\langle(1,2)(3,4)\rangle$	1260	20	17
$\langle(1,2)(3,4)(5,6)\rangle$	676	22	17
$\langle(1,2)(3,4)(5,6)(7,8)\rangle$	508	13	8
$\langle(1,2,3)\rangle$	0	0	0
$\langle(1,2,3)(4,5,6)\rangle$	90	9	6
$\langle(1,2,3,4,5)\rangle$	0	0	0
$\langle(1,2,3,4,5,6,7)\rangle$	13	3	3
	5007	73	57

Table 5.1: The minimal overgroups of id in S_8

Notice that this list of 57 designs still contains isomorphic pairs. Imagine, e.g., the automorphism group of a design contains conjugates of more than one minimal subgroup of S_8. Then this design is enumerated for each of these groups. Doing a final isomorphism test for all 57 representatives of the 8 transversals in Table 5.1 we get 39 designs which have a nontrivial automorphism group. In Table 5.2, for each occuring automorphism group A, we specify its order, the number of orbits on points, the number of orbits on blocks, the point rank, the block rank and the number of isomorphism classes of designs with this automorphism group. Call these $|A|, o_P, o_{\mathcal{B}}, r_p, r_{\mathcal{B}}, n_d$ respectively. In Table 5.3 a complete description of all these groups will be given (in terms of generators).

A	$\lvert A\rvert$	o_P	$o_{\mathcal{B}}$	r_P	$r_{\mathcal{B}}$	n_d
G_1	336	1	1	2	6	1
G_2	32	1	3	5	36	1
G_3	16	1	3	6	53	1
G_4	8	1	5	8	102	1
G_5	12	2	4	8	72	1
G_6	12	2	4	10	80	1
G_7	7	2	4	10	112	2
G_8	4	2	7	16	196	1
G_9	4	3	8	18	200	1
G_{10}	8	4	9	20	140	1
G_{11}	4	3	8	20	200	1

A	$\|A\|$	o_P	$o_{\mathcal{B}}$	r_P	$r_{\mathcal{B}}$	n_d
G_{12}	8	4	7	21	136	1
G_{13}	4	4	9	22	204	1
G_{14}	3	4	12	24	272	3
G_{15}	4	5	11	30	236	1
G_{16}	2	5	16	34	400	9
G_{17}	2	6	18	40	424	10
G_{18}	2	7	20	50	464	2

Table 5.2: The designs with parameters $(8, 28, 14, 4, 6)$ having a nontrivial automorphism group

Remarks:

1) Note that in general it is not true that two designs having the same automorphism group also have the same point rank and block rank respectively. Even the number of point and block orbits may differ. However, in the table above exactly this happens.

2) Here and below we use sometimes the notation for groups as they appear in the library of GAP. For example, $G_{(16,9)}$ means the group # 9 in the catalogue of groups of order 16. In each such case we use the sign $(*)$.

$$
\begin{array}{lcll}
G_1 & = & \langle (1,2,3)(6,7,8), (1,3)(2,4)(5,6)(7,8), & = \ \mathrm{PGL}(2,7) \\
 & = & (2,3)(4,5)(6,7)\rangle & \\
G_2 & = & \langle (1,2)(3,4), (1,2)(7,8), (3,4)(5,6), & = \ G_{(32,46)}\ (*) \\
 & = & (1,3,5,7)(2,4,6,8)\rangle & \\
G_3 & = & \langle (1,2)(3,4)(5,6)(7,8), (3,5)(4,6), & = \ G_{(16,9)}\ (*) \\
 & = & (1,3,2,4)(5,7,6,8)\rangle & \\
G_4 & = & \langle (1,2)(3,5)(4,6)(7,8), (1,3,2,5)(4,8,6,7), & = \ Q_8 \\
 & = & (1,4,2,6)(3,7,5,8)\rangle & \\
G_5 & = & \langle (1,2)(3,4)(5,6), (2,3)(4,5)(7,8)\rangle & = \ D_6 \\
G_6 & = & \langle (1,2), (3,4,5,6,7,8)\rangle & = \ \mathbb{Z}_2 + \mathbb{Z}_6 \\
G_7 & = & \langle (1,2,3,4,5,6,7)\rangle & = \ \mathbb{Z}_7 + Id_1 \\
G_8 & = & \langle (1,3,2,4)(5,7,6,8)\rangle & = \ \mathbb{Z}_4 \times Id_2 \\
G_9 & = & \langle (1,2)(3,4)(5,6), (1,3)(2,4)(7,8)\rangle & = \ E_4 \\
G_{10} & = & \langle (1,2), (5,6)(7,8), (3,4)(7,8)\rangle & = \ E_8
\end{array}
$$

$$
\begin{array}{lllll}
G_{11} & = & \langle(1,3,2,4)(5,6)(7,8)\rangle & = & \mathbb{Z}_4 \\
G_{12} & = & \langle(1,2),(3,4),(1,3)(2,4)(7,8)\rangle & = & D_4 \\
G_{13} & = & \langle(1,2)(3,4)(5,6),(1,3)(2,4)\rangle & = & E_4 \\
G_{14} & = & \langle(1,2,3)(4,5,6)\rangle & = & \mathbb{Z}_3 \times Id_2 \\
G_{15} & = & \langle(1,2),(3,4)(7,8)\rangle & = & E_4 \cong \mathbb{Z}_2 + \mathbb{Z}_2 \times Id_2 \\
G_{16} & = & \langle(1,2)(3,4)(5,6)\rangle & = & \mathbb{Z}_2 \times Id_3 \\
G_{17} & = & \langle(1,2)(3,4)\rangle & = & \mathbb{Z}_2 \times Id_2 \\
G_{18} & = & \langle(1,2)\rangle & = & \mathbb{Z}_2 + Id_6
\end{array}
$$

Table 5.3

Combining Lemma 3.6 and Corollary 3.7 we can compute the number of designs with trivial automorphism group up to isomorphism.
Lemma 3.6 gives for each isomorphism type of designs with nontrivial automorphism group the size of the respective isomorphism class. The results of these computation are given in the next table. For each appearing order of automorphism groups the number and sizes of isomorphism classes are given:

order	index in S_8	# iso-classes	# labelled designs
336	120	1	120
32	1,260	1	1,260
16	2,520	1	2,520
12	3,360	2	6,720
8	5,040	3	15,120
7	5,760	2	11,520
4	10,080	5	50,400
3	13,440	3	40,320
2	20,160	21	423,360
		39	551,340

Table 5.4

Altogether there are 551,340 labeled designs with nontrivial automorphism group. Hence the number of labeled designs with trivial automorphism group is:

$$5,591,340 - 551,340 = 5,040,000.$$

Application of Corollary 3.7) gives the number of isomorphism types of designs with trivial automorphism group:

$$5,040,000/40320 = 125.$$

Therefore we get $125 + 39 = 164$ isomorphism classes of designs with parameters $(8, 28, 14, 4, 6)$. This coincides with the results of B. Schmalz in [Sch 92].
We are now after the determination of explicit representatives for all these designs. Having split the task into pieces we are left with more than 5 million design for which at first glance there is only a brute force attack to reduce them up to isomorphism.
However in this particular case it still is possible to do the complete constructive enumeration of the 125 designs within a few hours of computation time. Let us describe this now.
First of all it is not necessary to take into account all 5,040,000 labeled designs with trivial automorphism group. It is enough to have a list which contains a transversal of the isomorphism classes. A good reduction of the effort can be achieved by prescribing certain blocks. Let us assume we have $\mathcal{P} = \{1, 2, \ldots 8\}$. Then w.l.o.g. we can prescribe $\{1, 2, 3, 4\} \in \mathcal{B}$. Define

$$n_i \;=\; |\{B \in \mathcal{B} \mid |\{1, 2, 3, 4\} \cap B| = i\}|,$$

called the i-th intersection number of $\{1, 2, 3, 4\}$ with $\mathcal{B}$. Using Lemma 2.3 we get the following:

$$\begin{aligned} n_0 + n_1 + n_2 + n_3 &= 28 - 1 = 27, \\ n_1 + 2n_2 + 3n_3 &= 13 \cdot 4 = 52, \\ 2n_2 + 6n_3 &= 5 \cdot 12 = 60. \end{aligned}$$

Suppose $n_2 = 0$. Then

$$\begin{aligned} n_0 + n_1 + n_3 &= 27, \\ n_1 + 3n_3 &= 52, \\ 6n_3 &= 60. \end{aligned}$$

gives $n_3 = 10$, $n_1 = 22$ and $n_0 = -5$ which is clearly impossible. Thus, $n_2 > 0$ and we can prescribe another block which intersects $\{1, 2, 3, 4\}$ in exactly two points. W.l.o.g let this be $\{1, 2, 5, 6\}$.
Among the designs of the list $\mathcal{L}_1$, 968,868 fulfill these two conditions. Call them $\tilde{\mathcal{L}}_1$. $\tilde{\mathcal{L}}_1$ still includes designs with non-trivial automorphism group. We remove these designs thereby geting a new list $\overline{\mathcal{L}}_1$. Finally the following algorithm is applied to $\overline{\mathcal{L}}_1$ in order to partition it into classes of isomorphic designs:
Input:

1) The reduced list $\overline{\mathcal{L}}_1$ of labeled designs having a trivial automorphism group (reduced in the above mentioned sense). $\overline{\mathcal{L}}_1$ be indexed by the set $\{0, 1, \ldots, |\overline{\mathcal{L}}_1|\}$.

2) The number n of all isomorphism classes of designs having a trivial automorphism group (this is known from the previous computations)

Output:
A list $\mathcal{O}$ of the isomorphism classes of the designs from $\overline{\mathcal{L}}_1$.
Variables:
An index variable i in order to loop over $\overline{\mathcal{L}}_1$.
Init:
$\mathcal{O} := \emptyset$;
$i := 0$;
Program:

```
WHILE (|𝒪| < n) DO
      REPEAT
            𝔇 := L̄₁[i]; i := i + 1;
      UNTIL (∄O ∈ 𝒪 : 𝔇 ∈ O);
      O := Orb(S_v, 𝔇);
      𝒪 := 𝒪 ∪ {O};
OD;
RETURN 𝒪;
```

This simple algorithm computes all orbits of designs and has at a certain point the whole list of designs in memory. For greater v's this may not be feasible. Using combinatorial invariants such as e.g. intersection arrays one can prepartition the list of designs with trivial automorphism group. Then the above described algorithm has to be applied to each equivalence class. This may reduce the necessary effort considerably.

6 Small designs revisited

In the following we test our strategy on designs with small parameters. Although most of these computations were done before e.g. by A.V.Ivanov (see [Iva 85]), we think for several reasons that it is worthwhile to repeat them.

1) Using a different algorithmic approach we can reconfirm the correctness of the previous results.

2) Doing this we can also test the feasibility of our strategy and find out what are its limits.

3) Last but not least, our strategy has the advantage that it delivers the complete automorphism group of each design. Thus possibly interesting actions of groups can be found that are of independent interest.

Following are some informations about the designs for the various parameter sets (where $2b \leq \binom{v}{k}$). The designs themselves are given in the supplement. For each design $\mathfrak{D}$, we specify the automorphism group, its order, the number of orbits on points, the number of orbits on blocks, the point rank and the block rank. Call these $A, |A|, o_P, o_\mathcal{B}, r_p, r_\mathcal{B}$ respectively. Only data for designs with nontrivial automorphism group is given.

$\mathfrak{D}$	A	$\|A\|$	o_P	$o_\mathcal{B}$	r_P	$r_\mathcal{B}$
$(6,10,5,3,2)$						
$\mathfrak{D}_1$	A_5	60	1	1	2	3
$(7,7,3,3,1)$ the Fano plane						
$\mathfrak{D}_1$	PSL(3, 2)	168	1	1	2	2
$(7,14,6,3,2)$						
$\mathfrak{D}_1$	AGL(1, 7)	42	1	1	2	6
$(8,14,7,4,3)$ 4 designs						
$\mathfrak{D}_1$	AGL(3, 2)	1344	1	1	2	3
$\mathfrak{D}_2$	$\mathbb{Z}_7 \rtimes \mathbb{Z}_3$	21	2	2	6	12
$\mathfrak{D}_3$	$\mathbb{Z}_2 \times S_4$	48	2	3	7	16
$\mathfrak{D}_4$	A_4	12	3	4	12	28
$(9,12,4,3,1)$						
$\mathfrak{D}_1$	AGL(2, 3)	432	1	1	2	3
$(9,24,8,3,2)$ 13 designs						

$\mathfrak{D}$	A	$\|A\|$	o_P	$o_{\mathcal{B}}$	r_P	$r_{\mathcal{B}}$
$\mathfrak{D}_1$	$S_3 \times \mathbb{Z}_3$	18	1	2	6	34
$\mathfrak{D}_2$	$\mathrm{AGL}(2,3)$	80	2	2	7	18
$\mathfrak{D}_3$	$\mathbb{Z}_8$	8	2	3	11	72
$\mathfrak{D}_4$	D_3	6	2	4	14	96
$\mathfrak{D}_5$	$\mathbb{Z}_6$	6	2	4	15	96
$\mathfrak{D}_6$	$\mathbb{Z}_6$	6	2	5	15	102
$\mathfrak{D}_7$	D_4	8	3	4	16	80
$\mathfrak{D}_8$	$\mathbb{Z}_2$	2	5	12	41	288
$\mathfrak{D}_9$	$\mathbb{Z}_2$	2	6	14	45	296
$\mathfrak{D}_{10}$	$\mathbb{Z}_2$	2	6	14	45	296

$(9, 18, 8, 4, 3)$ 11 designs

$\mathfrak{D}_1$	$E_9 \rtimes 2D8\,(*)$	144	1	1	2	6
$\mathfrak{D}_2$	$G_{(32,46)}\,(*)$	32	2	3	8	21
$\mathfrak{D}_3$	$QD16\,(*)$	16	2	3	8	30
$\mathfrak{D}_4$	$\mathbb{Z}_9$	9	1	2	9	36
$\mathfrak{D}_5$	E_8	8	5	7	29	68
$\mathfrak{D}_6$	D_8	8	3	6	15	60
$\mathfrak{D}_7$	$\mathbb{Z}_6$	6	2	4	15	60
$\mathfrak{D}_8$	$\mathbb{Z}_2$	2	6	12	45	180
$\mathfrak{D}_9$	$\mathbb{Z}_2$	2	6	12	45	180

7 Other examples

In this section we are going to present a few new results. As a rule the motivation of the selection of a considered parameter set was to increase the number of known designs drastically (in a part of the cases from a few known to thousands of new designs). We were either working with a prescribed permutation group (H, Ω) or with all overgroups (H, Ω) in $S(\Omega)$ of a prescribed permutation group (H_o, Ω).
In some cases the information about the actions of the automorphism groups on points and blocks was omitted because of the huge amount of designs for this group in order to keep the tables small.

A	$\|A\|$	$o_{\mathcal{B}}$	r_P	$r_{\mathcal{B}}$	n_d

A	$\|A\|$	$o_{\mathcal{B}}$	r_P	$r_{\mathcal{B}}$	n_d
$(11, 55, 20, 4, 6)$, $H = \mathbb{Z}_{11}$; 66 designs					
PSL$(2, 11)$	660	1	2	8	1
AGL$(1, 11)$	110	1	2	30	2
$\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$	55	1	3	55	1
D_{11}	22	4	6	142	1
$\mathbb{Z}_{11}$	11	5	11	275	61

$(11, 110, 40, 4, 12)$, $H = \mathbb{Z}_{11}$; 5,759 designs					
PSL$(2, 11)$	660	1	2	22	1
AGL$(1, 11)$	110	1	2	110	2
AGL$(1, 11)$	110	2	2	120	1
$\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$	55	2	3	220	5
D_{11}	22	7	6	558	5
D_{11}	22	8	6	568	2
$\mathbb{Z}_{11}$	11	10	11	1100	5,743

$(13, 78, 30, 5, 10)$, $H > \mathbb{Z}_{13}$; 39 designs					
AGL$(1, 13)$	156	1	2	42	2
$\mathbb{Z}_{13} \rtimes \mathbb{Z}_6$	78	1	3	78	2
$\mathbb{Z}_{13} \rtimes \mathbb{Z}_4$	52	4	4	128	1
$\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$	39	2	5	156	30
D_{13}	26	3	7	234	4

$(14, 91, 26, 4, 6)$, $H = \mathbb{Z}_{14}$; 10,196 designs					
PSL$(3, 2) \times \mathbb{Z}_2$	336	3	3	51	1
$\mathbb{Z}_7 \rtimes S_3$	42	4	6	213	12
D_{14}	28	7	8	322	2
D_{14}	28	7	8	324	4
D_{14}	28	7	8	330	1
$\mathbb{Z}_{14}$	14	—	—	—	10,176

$(14, 91, 39, 6, 15)$, $H > \mathbb{Z}_{14}$; 294 designs					
PGL$(2, 13)$	2184	1	2	8	1
PSL$(2, 3) \times \mathbb{Z}_2$	336	2	3	36	1
PSL$(2, 3) \times \mathbb{Z}_2$	336	3	3	49	1
$\mathbb{Z}_{14} \rtimes \mathbb{Z}_3$	42	3	6	199	238

A	$\|A\|$	$o_{\mathcal{B}}$	r_P	$r_{\mathcal{B}}$	n_d
D_{14}	28	5	8	302	12
D_{14}	28	6	8	312	24
D_{14}	28	7	8	322	11
D_{14}	28	7	8	324	4
D_{14}	28	7	8	334	1
D_{14}	28	8	8	356	1

$(15, 42, 14, 5, 4)$, $H = (\mathbb{Z}_7 \times id_2) + id_1$;
3,664 designs

A	$\|A\|$	o_P	$o_{\mathcal{B}}$	r_P	$r_{\mathcal{B}}$	n_d
A_7	2520	1	1	2	4	1
$AGL(3,2)$	1344	2	2	6	13	1
$E_8 \rtimes F_{21}$	168	2	2	7	18	1
${\mathbb{Z}_2}^3 \rtimes \mathbb{Z}_7$	56	2	2	11	44	4
$\mathbb{Z}_7 \rtimes \mathbb{Z}_3$	21	3	4	17	90	21
$\mathbb{Z}_2 \times \mathbb{Z}_7$	14	3	6	21	144	1
$\mathbb{Z}_2 \times \mathbb{Z}_7$	14	2	4	17	140	3
D_7	14	2	3	17	126	2
$\mathbb{Z}_2 \times \mathbb{Z}_7$	14	2	3	17	126	81
$\mathbb{Z}_7$	7	3	6	33	252	3549

$(15, 105, 35, 5, 10)$, $H > \mathbb{Z}_{15}$; 4,751 designs

A	$\|A\|$	$o_{\mathcal{B}}$	r_P	$r_{\mathcal{B}}$	n_d
$S_3 \times D_5$	60	4	6	198	3
$S_3 \times D_5$	60	4	6	206	2
$S_3 \times D_5$	60	4	6	218	2
$D_5 \rtimes \mathbb{Z}_4$	60	3	5	191	3
$S_3 \times D_5$	60	3	5	191	2
$(D_5 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	60	3	5	191	2
$D_5 \rtimes \mathbb{Z}_4$	60	3	6	192	5
$S_3 \times D_5$	60	3	6	192	4
$(D_5 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	60	3	6	192	3
$D_5 \rtimes \mathbb{Z}_4$	60	4	5	221	1
$(D_5 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	60	4	5	221	1

A	$\|A\|$	$o_{\mathcal{B}}$	r_P	$r_{\mathcal{B}}$	n_d
$(D_5 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	60	4	6	198	3
$D_5 \rtimes \mathbb{Z}_4$	60	4	6	206	2
$D_5 \rtimes \mathbb{Z}_4$	60	4	6	218	2
$S_3 \times D_5$	60	4	6	222	1
$S_3 \times \mathbb{Z}_5$	30	4	10	370	924
$S_3 \times \mathbb{Z}_5$	30	5	10	390	1195
$S_3 \times \mathbb{Z}_5$	30	6	10	430	78
$D_5 \times \mathbb{Z}_3$	30	4	9	369	752
$D_5 \times \mathbb{Z}_3$	30	5	9	381	1614
$D_5 \times \mathbb{Z}_3$	30	6	9	405	114
$D_5 \times \mathbb{Z}_3$	30	7	9	441	6
D_{15}	30	5	8	372	29
D_{15}	30	6	8	380	1
D_{15}	30	7	8	392	2
$(17, 68, 16, 4, 3)$, $H = \mathbb{Z}_{17}$; 165 designs					
$AGL(1, 17)$	272	1	2	20	1
$\mathbb{Z}_{17} \rtimes \mathbb{Z}_8$	136	1	3	36	1
$\mathbb{Z}_{17}$	17	4	17	272	163

8 Bibliographical comments and remarks

In this section we are giving some bibliographical and historical comments on the problem of the enumeration of designs with the parameters which are considered in our paper. By no means we are pretending to submit a complete bibliographical survey. More exhaustive references may be traced e.g. from [BetJL 93], [MatR 96] and other sources.

The main portion of information is organized with the aid of Table 8.1. We compare the numbers of designs which were known to A.V.Ivanov, and those which appear in [MatR 96], with our results. An exact number x in a column of the following table means the claim that all designs are known (to the corresponding authors), and there are up to isomorphism exactly x designs. The sign $\geq x$ means that there are known at least x pairwise non-isomorphic designs. In the references we try to give the most relevant publication (for a prescribed set of parameters), including one of earliest description of a design (designs). The sign "?" means that the corresponding problem was not solved. An empty item means that there was no information on this subject. Some extra comments and remarks are given below.

#	v	b	r	k	λ	Ivanov's list	CRC list	Our results	References
1	6	10	5	3	2	1	1	1	[Nan 46a]
2	7	7	3	3	1	1	1	1	classical
3	7	14	6	3	2	1	1	1	classical
4	8	14	7	4	3	4	4	4	[Nan 46b]
5	8	28	14	4	6	?	?!	164	[Emc 29], [GroR 81]
6	9	12	4	3	1	1	1	1	classical
7	9	24	8	3	2	13	?!	13	[Iva 80]
8	9	18	8	4	3	11	11	11	[Gib 76]
9	9	36	12	3	3	≥ 330	?!	?	[HarCI 87]
10	11	55	20	4	6		≥ 1	≥ 66	[Tak 62]
11	11	110	40	4	12		≥ 1	$\geq 5{,}759$	Double [Tak 62]
12	13	78	30	5	10		≥ 31	≥ 39	Double [TonR 82], [Hal 86]
13	14	91	26	4	6		≥ 4	$\geq 10{,}196$	[MenH 72], [Han 75]
14	14	91	39	6	15		≥ 1	≥ 294	[Han 75]
15	15	42	14	5	4		≥ 207	$\geq 3{,}664$	[Han 61], [Han 75], [Neu 84], [MatR 89], [HobB 90],[KhoNT 95],
16	15	105	35	5	10		≥ 1	$\geq 4{,}751$	[Abe 94]
17	17	68	16	4	3		≥ 1	≥ 165	[Tak 62]

Table 8.1

8.1. The authors of [MatR 96] give in their parameter tables the numbers of all designs with a prescribed parameter set (including designs with repeated blocks) known to them. In rows 1–4, 6, 8 of our Table 8.1 the numbers of known designs in [MatR 96] coincide with the numbers in our table. The sign "?!" in a few rows shows that the exact number of designs without repeated blocks is not traced out from [MatR 96] (nevertheless it may be found in some of the papers which are referred there). The numbers in rows 10–17 give, as a rule, an estimation of the number of designs with or without repeated blocks. In cases when only one design is known we did not always try to realize from the literature if this design is with or without repeated blocks.

8.2. The reference "classical" in Table 8.1 means that the question was treated already in the XIX-th century, however we are not making here an attempt to cite the first paper where a corresponding design was discovered or characterized.

8.3. The word "double" in rows 11, 12 means that the only previously known designs with the parameters $(v, 2b_1, 2r_1, k, 2\lambda_1)$ were obtained as a union of two designs (perhaps equal) with the same parameters $(v, b_1, r_1, k, \lambda_1)$. This implies, for example, for row 11 that the only known design had repeated blocks. Therefore for this set $(11, 110, 40, 4, 12)$ the existence of designs without repeated blocks follows from our computations.

8.4. References in Table 8.1 mostly are collected from survey papers about designs especially from the CRC paper by R.Mathon & A.Rosa. As we shall show below, some of the designs which have appeared through our enumeration were known before. This holds, in particular, for designs with "high symmetry".

8.5. Row 15 corresponds to a parameter set which was discussed many times in the literature in different contexts. During a long time only a few such designs were known: among them are two designs discovered by H.Hanani ([Han 61], [Han 75]) and one with large automorphism group (see below) which appears from the famous Witt design $S(5, 8, 24)$, see, e.g., [Neu 84]. R.Mathon and A.Rosa described in [MatR 89] 85 1-rotational 2–$(15, 5, 4)$ designs. On the next stage S.A.Hobart and W.G.Bridges discovered in [HobB 90] 15 new designs with the same parameters. In [KhoNT 95] the amount of known designs was increased from 103 to 207. Thus our current bound is more than 10 times larger.

8.6. Row 9 was the only obstacle for us to complete the constructive enumeration of all small designs (with $v \leq 9$). We were not able to get in a reasonable time (say 10–15 hours) the list of all labeled designs. Hence it demonstrates in a sense some restrictions of our attack with the aid of the current version of DISCRETA (These restrictions will be discussed with more details in the next section).

Nevertheless, it is worthwhile to mention that this case is really comparably difficult. For example, A.V.Ivanov, using his algorithm during 4.5 hours (see [Iva 83]), found 330 pairwise non-isomorphic designs but did not finish an exhaustive search for all designs. In fact, the problem was settled later on by Ivanov jointly with J.J.Harms and Ch.J.Colbourn using more sophisticated techniques than standard orderly generation (see [HarCI 87]) for details). Altogether there are 332 designs.

8.7. One of the subordinate results of our enumeration is a reasonably large amount of block-transitive designs. Such designs are of a substantial interest, see, e.g., [Cam 94]. Block transitive designs with an imprimitive automorphism group are comparatively rare objects (see [DelD 89]). Altogether we constructed 22 block-transitive designs, all of them have a primitive automorphism group. At least 6 of them (which correspond to rows 1–6) are regarded as classical, therefore we do not discuss them with more details. All other such designs are exposed below from some different points of view.

8.8. The unique block-transitive design $\mathfrak{D}$ with the parameter set in row 15 is certainly of a special interest. In implicit manner it was described a few times, see references in [HobB 90]. An explicit description by Hobart-Bridges refers to the famous Witt design $S = S(5, 8, 24)$: take a block B of S and choose two points x, y such that $x \in B$ and $y \notin B$. Then the points of $\mathfrak{D}$ are the points of S outside of $B \cup \{y\}$, while the blocks of $\mathfrak{D}$ are those blocks of S which contain x and y, and intersect B in two points. It was shown in [HobB 90] that $\mathrm{Aut}(\mathfrak{D}) = A_7$.
Hereby we suggest a new interpretation of the design $\mathfrak{D}$ which may be of some independent interest.
Let F be a labeled design with the point set $\Omega = \{1, 2, \ldots, 7\}$ which is isomorphic to the Fano plane, the unique $(7, 7, 3, 3, 1)$-design. We know that $\mathrm{Aut}(F) = \mathrm{PSL}(3, 2)$. This follows that there exist 30 different designs in the isomorphism class of F and that A_7 has two orbits of length 15 on this class. Let P be one of such orbits. It is easy to check (see, e.g., [Llo 95]) that all other 14 Fano planes from P have exactly one common block with F. Let us select $i \in \Omega$ and let us define a graph Γ_i with the vertex set P such that two distinct planes $F_1, F_2 \in P$ are joined by an edge if their common block does not contain i. We may translate the definition of Γ_i into other terms. Each Fano plane $F \in P$ includes exactly three lines through the point i, Thus these lines define a partition $\tilde{F}$ of the set $\Omega \setminus \{i\}$ into three 2-element

subsets. Now two partitions $\tilde{F}_1$, $\tilde{F}_2$ are joined by an edge in the new graph $\tilde{\Gamma}_i$ if and only if they do not have a common pair. It is evident that the graph $\tilde{\Gamma}_i$ is isomorphic to Γ_i.

Now we will exploit a famous result about the properties of number six (see, e.g., [CamL 91]). One of the reformulations of this result is that each graph $\tilde{\Gamma}_i$ is isomorphic to the triangular graph T_6. (We recall that the vertices of T_6 are all 2-element subsets of a 6-element set, two vertices are joined by an edge, if the corresponding pairs have a common point.). The description of cliques in T_6 is well-known: in particular it has six 5-cliques, each clique consists of all pairs which include a prescribed point.

The seven graphs Γ_i, $i \in \Omega$ define altogether $7 \cdot 6 = 42$ cliques. We consider an incidence structure $\mathfrak{D} = (P, \mathcal{B})$ with the point set P and the above mentioned 42 blocks of size 5. This structure $\mathfrak{D}$ is invariant with respect to the action (A_7, P) which is well-known to be 2-transitive. Therefore $\mathfrak{D}$ is a 2-design.

In principle, our description may also help to show that $\mathrm{Aut}(\mathfrak{D}) = A_7$, that is to confirm once more this equality without the use of a computer.

It is worthwhile to mention that some other exceptional combinatorial structures defined in terms of different labelings of the Fano plane will be considered in [JonKL] and [KliR].

8.9. Description of two designs on 11 points with $\mathrm{PSL}(2, 11)$.

Let $\mathfrak{D}$ be the unique biplane on 11 points, see, e.g. [HugP 85]. Then $\mathfrak{D}$ has parameters $(11, 11, 5, 5, 2)$ and $\mathrm{Aut}(\mathfrak{D}) = \mathrm{PSL}(2, 11)$. Let $\mathcal{P}$ be the set of points of $\mathfrak{D}$ and define $\mathcal{B}$ to be the set of all 4-element sets X of points from $\mathcal{P}$ such that there exists a block Y of $\mathfrak{D}$ with $X \subset Y$. Then $(\mathcal{P}, \mathcal{B})$ is a design with parameters $(11, 55, 20, 4, 6)$ and its automorphism group is $\mathrm{PSL}(2, 11)$.

Let $\mathcal{B}$ be the set of blocks of the previously constructed $2-(11, 4, 6)$ design. Let $\mathcal{P}$ be the set of points of this design. Define $\tilde{\mathcal{B}}$ to be the set of all 4-element subsets of $\mathcal{P}$ that are not expressible as the symmetric difference of two blocks from $\mathcal{B}$. Then $\tilde{\mathcal{B}}$ has 110 elements and $(\mathcal{P}, \tilde{\mathcal{B}})$ forms a design with parameters $(11, 110, 40, 4, 12)$ having the desired automorphism group $\mathrm{PSL}(2, 11)$.

8.10. Description of a design on 14 points with $\mathrm{PGL}(2, 13)$.

Consider $F = GF(13)$. Let $B_1 = \{1, 3, 9\}$ be the set of all elements from F^* which can be expressed in the form x^4 for a convenient $x \in F^*$. Define $B_2 := 2 \cdot B_1 = \{2 \cdot x \mid x \in B_1\}$ and $B := B_1 \cup B_2$. Finally we associate to B the set $\tilde{B} = \{(1, x) \mid x \in B\}$ of the canonical representatives of the corresponding projective points over F ($\tilde{B}$ consists of 6 elements). Then the orbit of $\tilde{B}$ under $\mathrm{PGL}(2,13)$ is the block-set of the desired design.

8.11. Description of the unique block-transitive $(9,4,3)$ design.
Let $\mathfrak{D} = (\mathcal{P}, \mathcal{B})$ be the affine plane of order 3. Let $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$ denote the four parallel classes of $\mathfrak{D}$. Consider some fixed pairing of the parallel classes - say $\{\mathcal{B}_1, \mathcal{B}_2\}$ and $\{\mathcal{B}_3, \mathcal{B}_4\}$. For each of these pairs $\{\mathcal{B}_i, \mathcal{B}_k\}$ we define a set of blocks as follows: let $A_1 \in \mathcal{B}_i$, $A_2 \in \mathcal{B}_j$. Then $A_1 \cup A_2$ contains 5 elements. Define the new block A to be $\mathcal{P} \setminus (A_1 \cup A_2)$. This way we get for each pair $3 \cdot 3 = 9$ blocks - altogether 18. These 18 blocks form the desired design.

8.12. All other non-classical block-transitive designs in our list have as automorphism group some subgroup $\mathrm{AGL}(1,p)$ for a suitable prime number p. As point set we will always consider the respective Galois field $GF(p)$. Then the set of blocks for each design can be described by one base block and a multiplier. All other blocks of the design can be obtained from the base block by applying the multiplier or by cyclical shifting. More exactly, application of a multiplier x means the use of the cyclic subgroup $\langle x \rangle$ of F^*. In order to be able to distinguish non isomorphic designs one can use the following simple invariant:
Let p be an odd prime number. Let $F = GF(p)$ be the Galois field on p elements such that $F = \{0, 1, \ldots, p-1\}$. Denote $\overline{F} = \left\{0, 1, \ldots, \frac{p-1}{2}\right\}$. Let M be any k-element subset of F. Define the following complete coloured graph $\Gamma_p(M)$ with the vertex set M and colours of edges $d \in \overline{F}$, such that an edge $\{x, y\}$ receives colour d if and only if $y - x \in \{d, -d\}$. We will call this graph *difference-cgr*.
It can be observed easily that

- for any $a \in F^*$: $\Gamma_p(M) \cong \Gamma_p(a \cdot M)$,
- for any $a \in F$: $\Gamma_p(M) \cong \Gamma_p(a + M)$.

This follows that in a design which has a subgroup of $\mathrm{AGL}(1,p)$ as block-transitive automorphism group the difference-cgrs of the blocks are all pairwise isomorphic. So, if we have two designs $\mathfrak{D}_1$ and $\mathfrak{D}_2$ given by the base blocks B_1 and B_2 and it is known that the automorphism groups of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are subgroups of $\mathrm{AGL}(1,p)$, then

$$\Gamma_p(B_1) \ncong \Gamma_p(B_2) \Longrightarrow \mathfrak{D}_1 \ncong \mathfrak{D}_2.$$

Let us use this observation for the remaining cases.
$(11, 55, 20, 4, 6)$: Take 3 as multiplier. The base blocks $\{0, 1, 2, 3\}$, $\{0, 1, 3, 4\}$ and $\{0, 1, 2, 5\}$ give the three desired block-transitive designs.

$(11, 110, 40, 4, 12)$: Take 2 as multiplier. The base blocks $\{0, 1, 2, 4\}$ and $\{0, 1, 2, 5\}$ yield the two desired block-transitive designs.

$(13, 78, 30, 5, 10)$: Take 6 as multiplier. The base blocks $\{0, 1, 2, 3, 4\}$, $\{0, 1, 2, 3, 8\}$ give the two designs with $\mathrm{AGL}(1, 13)$ as automorphism group. Take 4 as multiplier. The base blocks $\{0, 1, 2, 3, 7\}$ and $\{0, 1, 2, 4, 7\}$ yield the other two block-transitive designs.

$(17, 68, 16, 4, 3)$: Take 3 as multiplier. The base blocks $\{0, 1, 4, 5\}$ and $\{0, 1, 3, 15\}$ yield the two desired block-transitive designs.

9 Discussion

9.1. First, we would like to stress that the use of computers in design theory has a long-standing history and may be classified into a lot of different interesting variations. The survey [GraG 88] and the book [Alg 85] contain a lot of information on this subject. An account of more fresh results may be found in [Gib 96].
The orderly generation approach and the Kramer-Mesner method are in a sense two extreme points in a wide spectrum of different computer techniques used in design theory. One of our goals in this paper was to compare these opposite approaches, to find lines of their interrelations and to point out some opportunities of their mutual influence and simultaneous development.

9.2. As it was mentioned above, DISCRETA was created for the search of t-(v, k, λ) designs invariant with respect to a prescribed permutation group. As a rule, DISCRETA may be successfully used when the number of columns in the KM-matrix is limited by a few hundreds. Nevertheless, for a number of concrete problems we were successful to use DISCRETA even with about a thousand columns in the KM-matrix. In cases when (H, Ω) is sufficiently large, the entries of the KM-matrix also have a large variation, the latter fact usually enforces speed of the finding of new solutions by the used solvers.
Our experience shows that DISCRETA works sufficiently good for small groups

also, in case when small means that $|H| \sim |\Omega|$. In such a case the number of columns in the KM-matrix is approximately $\frac{1}{v}\binom{v}{k}$.
So that, if say $k = 4$, then this bound gives value 140 for $v = 17$ and 506 for $v = 25$; thus for $k = 4$ the case $v = 17$ seems really to be on the limit of the current opportunities of DISCRETA.

9.3. If H is the identical subgroup of $S(\Omega)$ then the use of DISCRETA may in a sense be regarded as a flippant computational experiment: in this case all entries of the KM-matrix belong to the set $\{0, 1\}$ and the solver has in fact to fulfill the process of the search of all possible designs. In other words by the routine brute force the solver has to look through all possible variants, not using isomorph rejection, symmetry of partially filled incidence matrices, etc. It is understandable that in this case the speed of DISCRETA will be essentially slower in comparison with the algorithms of orderly generation. Nevertheless, it was a pleasant surprise to observe that for almost all parameter sets with $v \leq 9$ we were able to manage the complete constructive enumeration by means of DISCRETA. This shows, in our opinion, the good abilities of the solvers to arrange an efficient backtracking in the course of exhaustive search of all solutions.

9.4. An analysis and a comparison of existing algorithms of orderly generation of designs is not one of our goals in this paper. However, it is worthwhile to stress that over the last years some new facilities were created by different research groups. As one of such examples we would like to mention the computer program DESY by C.Pietsch [Pie 93] which was created in Rostock, see also [Gru 96]. Another, very impressive example is provided by the recent paper [Lam 97].

9.5. Our paper in some extent belongs to the area of experimental mathematics. This means that our goal was not only to get some concrete results on constructive enumeration of designs, but also to investigate opportunities of existing computer packages from new viewpoints, to test their weak and strong features, to try some non-standard combinations of known techniques. Therefore, besides concrete enumerational results we got also a few ideas for some possible options for future development of computer packages. Some of them are listed below.

9.6. Let us consider the situation when a prescribed group (H, Ω) is sufficiently large, the number of $\{k\}$-orbits of (H, Ω) is much larger than a few hundreds, but most of the $\{k\}$-orbits have length larger than b. Then it is a worthwhile idea to enumerate only short $\{k\}$-orbits, that is those for which the setwise stabilizer has an index smaller that b in H. For such generation information about the subgroup structure of H (which will be managed via GAP and/or on a theoretical level) will play a crucial role.

9.7. The knowledge of the symmetry of the KM-matrix may help to fulfill isomorphic rejection of the produced solutions of the Diophantine system (see Theorem 2.6). Namely, a program may generate a transversal T of the set of orbits of the action of the automorphism group of the KM-matrix on the solution set. Such orbits will be (in general) obtained via a merging of suitable orbits of the action of $N = \mathfrak{N}_{S_v}(H)$ on the same solution set. Each solution $t \in T$ has to be split into a few solutions which will represent different orbits of N. After the isomorphism testing of all designs corresponding to the produced solution we will get the complete set of all non-isomorphic designs which are invariant with respect to (H, Ω).

9.8. In the case when we are looking for all designs with the prescribed parameters, that is when H is the identity subgroup of $S(\Omega)$, the automorphism group of the KM-matrix is exactly the symmetric group $S(\Omega)$. As it was shown by I.A.Faradžev and A.V.Ivanov the successful work of an orderly algorithm of constructive enumeration is based on the use of such notions as canonicity predicate, extension of predicate, automorphism of partially constructed incidence systems, inadmissible vectors and so on. We suppose that some analogues of these notions may be also used in the course of exhaustive search of all designs which are invariant with respect to a prescribed non-identical subgroups (H, Ω) of $S(\Omega)$. Such generalizations may create effective bridges between orderly generation and the Kramer-Mesner method.

9.9. In fact, some of the desired innovations which were discussed in the above subsections are now in the process of elaboration by the members of Bayreuth group. The genre of an "experimental" paper supposes by definition, that certain difficulties which were not overcome in the course of the current work over the paper may be effectively eliminated rather soon after the publication. For example, we hope that certain simple modification of our techniques will allow to fulfill the computations for the line 9 of Table 8.1 in a very convenient time.

By the way, this is the reason why we never protocoled the time of computations in this paper. Our experience of the work over current and other projects shows that the time parameter usually drastically decreased after a few iterations of polishing and developing a program.

9.10. Hence after some time the "dry remainder" of this paper will be just the confirmation of some old enumeration results and a number of new such results. In this context it is worthwhile to cite the following passage from [KolLT 90], see also [Lam 91] which we would like to call

Lam's Claim: "With the increasing use of computers in mathematics, the correctness of such (computer) "proofs" is very difficult to determine. We should borrow an idea from the physical sciences, where a new result is accepted only after it has been independently verified. The published results should contain enough information to allow independent cross checking of values, besides the final numbers. Internal consistency checking should also be used as much as possible even when it is expensive".

In sense of this claim we think that our results give a new evidence to some old results of constructive enumeration. Also we hope that in a future they may serve for the verification of new computer facilities for the constructive enumeration.

9.11. We will accomplish with the following anecdote which has only implicit relation to the main subject of the paper.

The number of main classes of 8×8 Latin squares (see [KolLT 90] for all details) was found by [ArlBGF 78] as 283,640. They also gave (in a supplement) the distribution of the orders of the automorphism groups of these squares. According to the prescribed distribution, one may find the number of reduced squares and compare it with the number T_8 which was obtained by M.B.Wells in [Wel 67]. It is claimed by the authors of [ArlBGF 78] that their result coincides with the result by Wells. In fact this is not true (as simple arithmetical computations show). The improved results by [KolLT 90] give the total number 283,657 of main classes and the distributions of the orders of the automorphism groups which really implies the Wells number T_8. Thus in [ArlBGF 78] 17 main classes of squares were missed. What is the reason? One of us had recently discussed this question with I.A.Faradžev [Far 97]. The following explanation sounds as sufficiently reasonable. As it was mentioned in [ArlBGF 78], the prolongation of all computations was about 60 hours. All computations were done during nights. The number of nights is now guessed to be between 10 and 20. After getting at EACH new main class the corresponding input to the total number of reduced classes was

automatically added. However the new point for a next computer start of the exhaustive search was each time input by hand. Due to some mistake in this input procedure, this intermediate class was taken into account via the computation of T_8 but was not reflected in the total catalogue of main classes. Unfortunately the authors of [ArlBGF 78] were too careless and did not arrange extra hand control of the obtained distribution of the orders of the automorphism groups. Therefore they did not catch the mistake.
We hope that the systematical following to the above mentioned Lam's claim will in a future help to avoid at least a part of similar mistakes in the course of constructive enumeration of combinatorial objects.

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