Symmetric Functions in MAGMA

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Overview

- Symmetric Functions
- Multiplication
- Plethysm
- Transition Matrices

Symmetric Functions

symmetric polynomial f

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- multivariate polynomial: $f \in \mathbb{Q}[x_1, x_2, x_3, \ldots, x_n]$

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symmetric function

- infinite number of variables but finite degree

monomial symmetric function

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- orbit of a monomial $x^I := x_1^{I_1} x_2^{I_2} \dots$ for $I \in \mathbb{N}^{\mathbb{N}}$of finite weight $= \Sigma I_i$

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- $m_I := Sym_{\mathbb{N}}(x^I)$
- $m_{0,2,0.1}(a,b,c,d) = a^2b+a^2c+a^2d+ab^2 \ldots = Sym_4(b^2d)$

Symmetric Functions

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- $(1 + a + b + c + \ldots)(a^2 + b^2 + c^2 + \ldots) = m_2 + m_3 + m_{2,1}$

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- $$e_3 = \begin{array}{|c|}\hline c \\ \hline b \\ \hline a \\ \hline \end{array} + \begin{array}{|c|}\hline d \\ \hline b \\ \hline a \\ \hline \end{array} + \dots \begin{array}{|c|}\hline d \\ \hline c \\ \hline b \\ \hline \end{array} + \begin{array}{|c|}\hline e \\ \hline c \\ \hline b \\ \hline \end{array} + \dots$$

generating function

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- e_I g.f. of standard tableaux of skew-shape $1^{I_1} \times 1^{I_2} \times \dots$

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generating function

- e_I g.f. of standard tableaux of skew-shape $1^{I_1} \times 1^{I_2} \times \dots$
- $$e_{3,2} = \begin{array}{cc} c & \\ b & \\ a & \\ & b \\ & a \end{array} + \begin{array}{cc} d & \\ b & \\ a & \\ & b \\ & a \end{array} + \dots \begin{array}{cc} d & \\ c & \\ b & \\ & b \\ & a \end{array} + \begin{array}{cc} e & \\ c & \\ b & \\ & b \\ & a \end{array}$$

Symmetric Functions

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- $h_k := \Sigma_{I \vdash k} m_I$ for $k \in \mathbb{N}$

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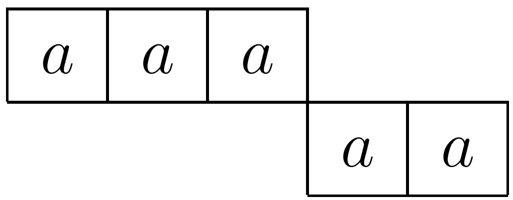
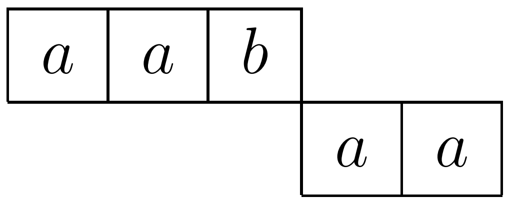
- $h_{3,2} =$ 

$+$ 

$+ \dots$

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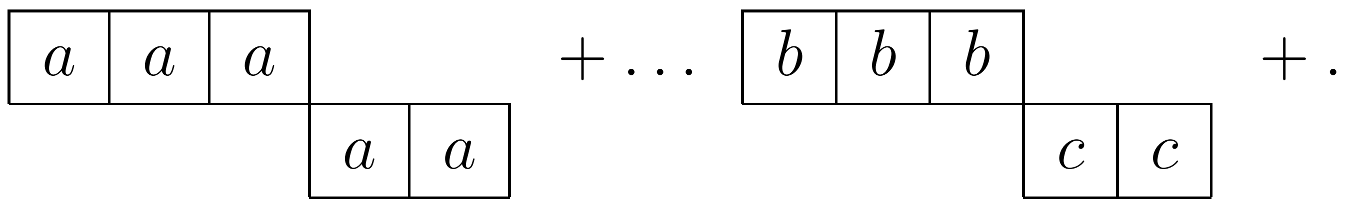
- $p_{3,2} =$ 

Symmetric Functions

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- $$p_{3,2} = \begin{array}{|c|c|c|cc}\hline a & a & a & \multicolumn{2}{c}{} \\ \hline \multicolumn{3}{c|}{} & a & \multicolumn{1}{|c|}{a} \\ \cline{4-5}\end{array} + \ldots \begin{array}{|c|c|c|cc}\hline b & b & b & \multicolumn{2}{c}{} \\ \hline \multicolumn{3}{c|}{} & c & \multicolumn{1}{|c|}{c} \\ \cline{4-5}\end{array} + \ldots$$
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Schur function

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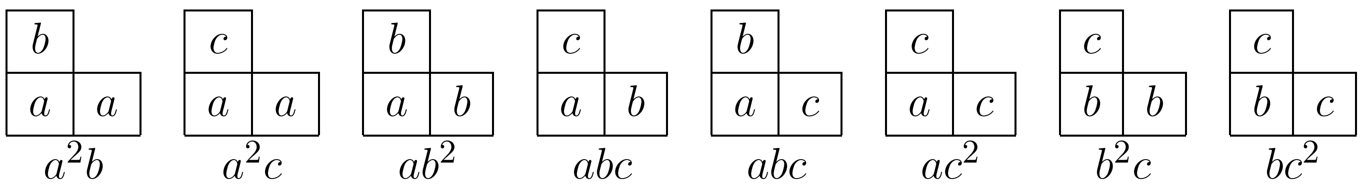
- $S_{2,1}(a,b,c) = S_{21}(A_3) =$



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b	c	b	c	b	c	c	c
$a\ a$	$a\ a$	$a\ b$	$a\ b$	$a\ c$	$a\ c$	$b\ b$	$b\ c$
a^2b	a^2c	ab^2	abc	abc	ac^2	b^2c	bc^2

$=a^2b+a^2c+ab^2+2abc+ac^2+b^2c+bc^2$

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Multiplication

algebra Λ of symmetric function

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- elementary symmetric $e_I \times e_J = \Sigma_K \ldots e_K$

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Multiplication

algebra Λ of symmetric function

- elementary symmetric $e_I \times e_J = \Sigma_K \ldots e_K \; = e_{I \cup J}$ trivial
- same for powersum, complete
- monomial symmetric $m_I \times m_J = \Sigma_K \ldots m_K$ simple

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Multiplication

product of Schur functions

- $S_I \times S_J = \Sigma_K c_{I,J,K} S_K$

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product of Schur functions

- $S_I \times S_J = \Sigma_K c_{I,J,K} S_K$
- Littlewood-Richardson Rule:

$c_{I,J,K} =$ number of some combinatorial objects

Multiplication

product of Schur functions

- $S_I \times S_J = \Sigma_K c_{I,J,K} S_K$
- Littlewood-Richardson Rule:

$c_{I,J,K} =$ number of some combinatorial objects

- useful for a single coefficient $c_{I,J,K}$

Multiplication

Littlewood Richardson rule

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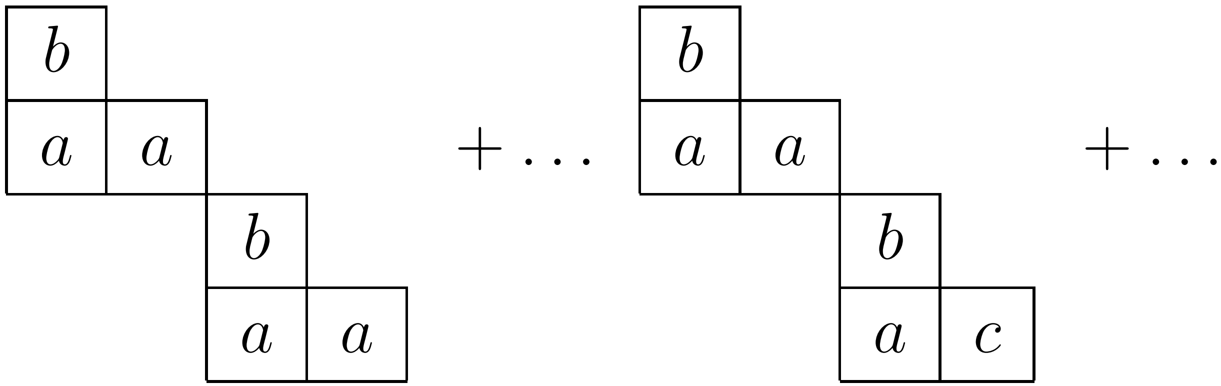

jeu de taquin

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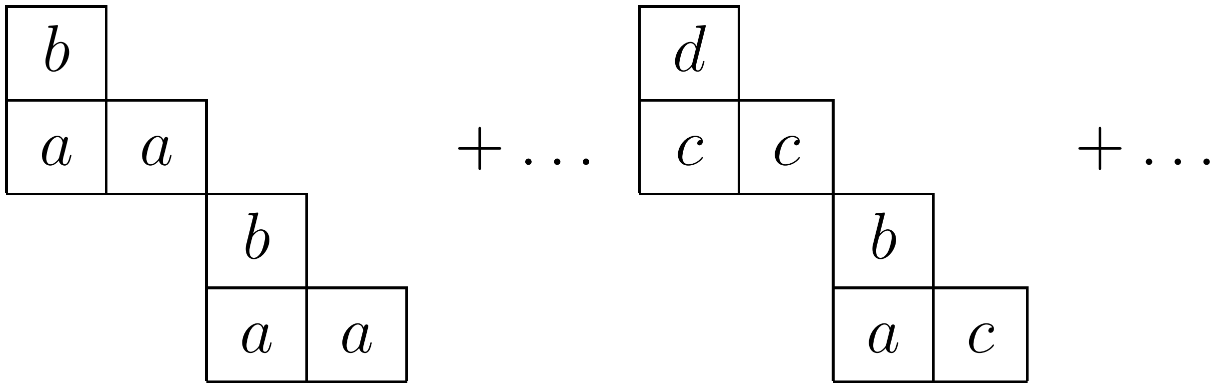
-



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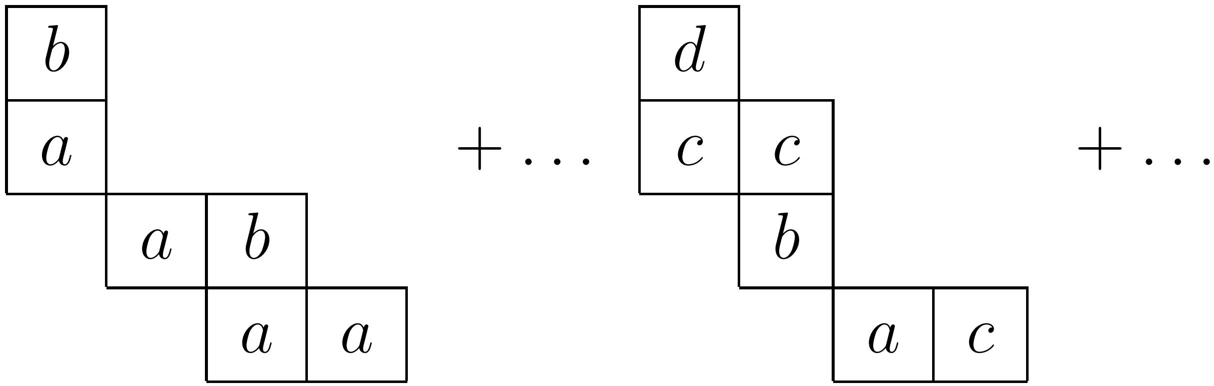
- 

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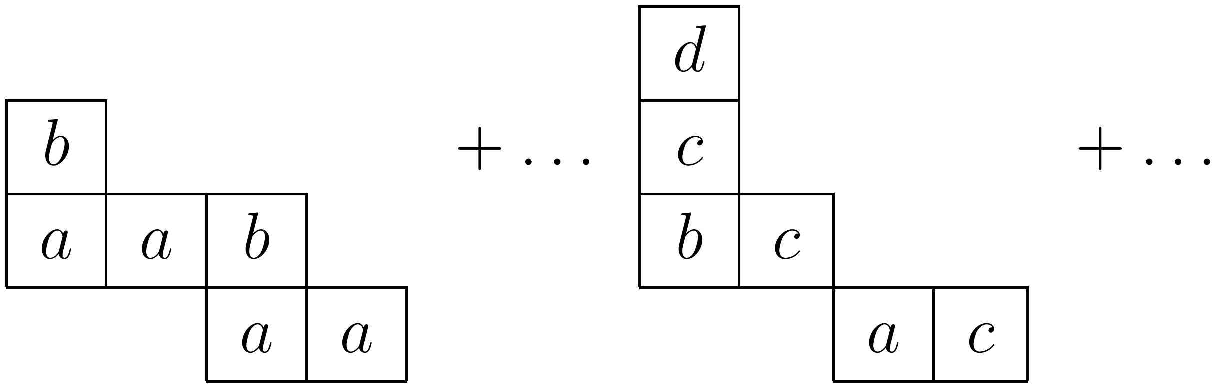
- 

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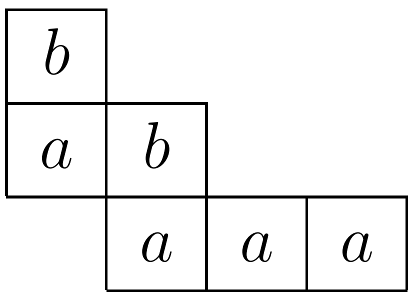
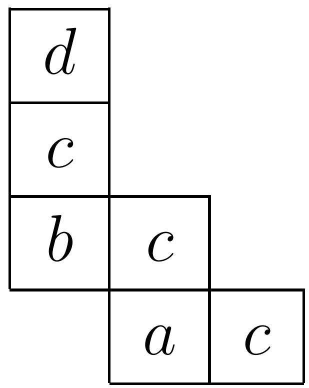
-  $+ \ldots$  $+ \ldots$

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Multiplication

jeu de taquin

- $S_{2,1} \times S_{2,1} =$
-

b	b		
a	a	a	a

$+ \ldots$

d		
c		
b		
a	c	c

$+ \ldots$

jeu de taquin

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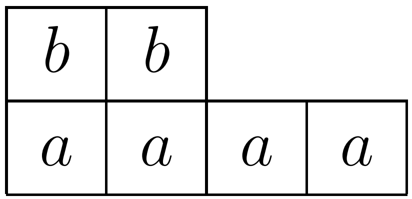
- 

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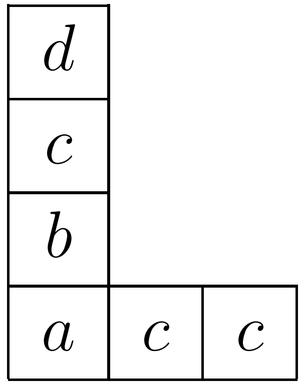

$+ \ldots$

$= S_{4,2} + \ldots\ldots\ldots + S_{3,1,1,1} + \ldots$

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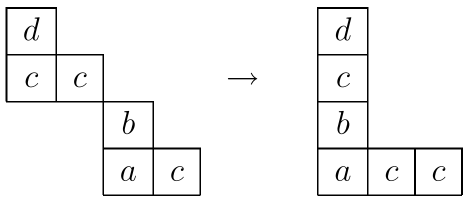
Littlewood Richardson rule

- combinatorial LR



Multiplication

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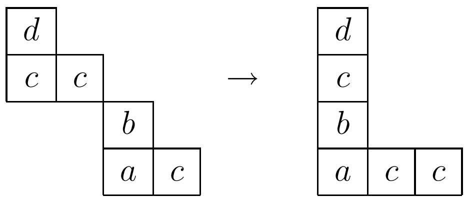
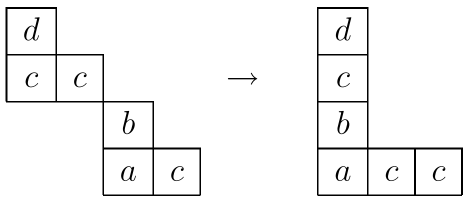


- polynomial LR

$$S_{21} \times S_{21} = \ldots + S_{3111} + \ldots$$

Littlewood Richardson rule

- combinatorial LR

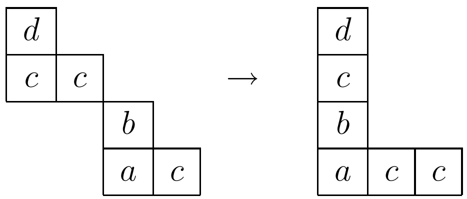


- polynomial LR
  $S_{21} \times S_{21} = \ldots + S_{3111} + \ldots$

- polynomial LR (fixed number of variables)
  $S_{21}(A_3) \times S_{21}(A_3) = \ldots + (S_{3111}(A_3) = 0) + \ldots$

Littlewood Richardson rule

- combinatorial LR (tableaux, non-commutative)



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Multiplication

Schubert polynomial

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Schubert polynomial

- multivariate $\in \mathbb{N}[x_1, x_2, \ldots, x_n]$
- labeled by permutations: X_w for $w \in Sym_n$
- homogeneous, degree $= l(w)$
- non-symmetric
- generalizes Schur polynomials

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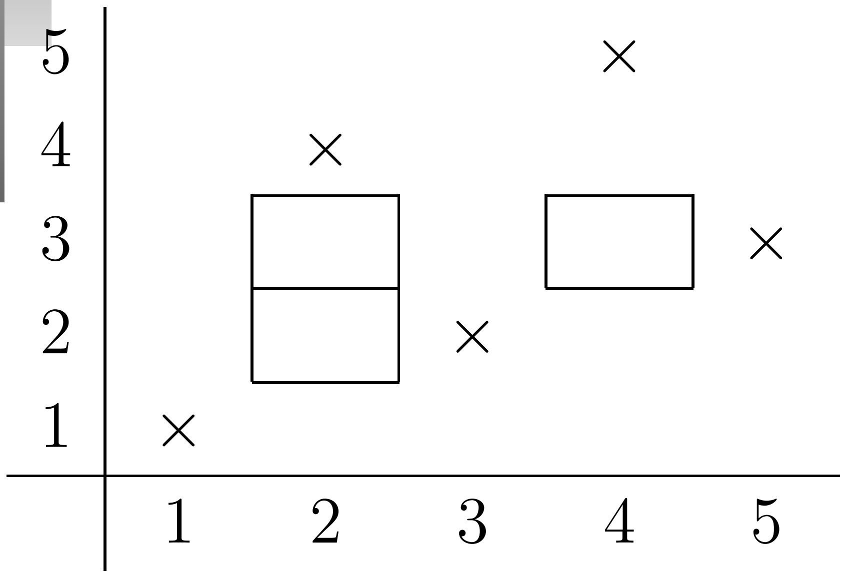
Schubert polynomial X_{13524}



Multiplication

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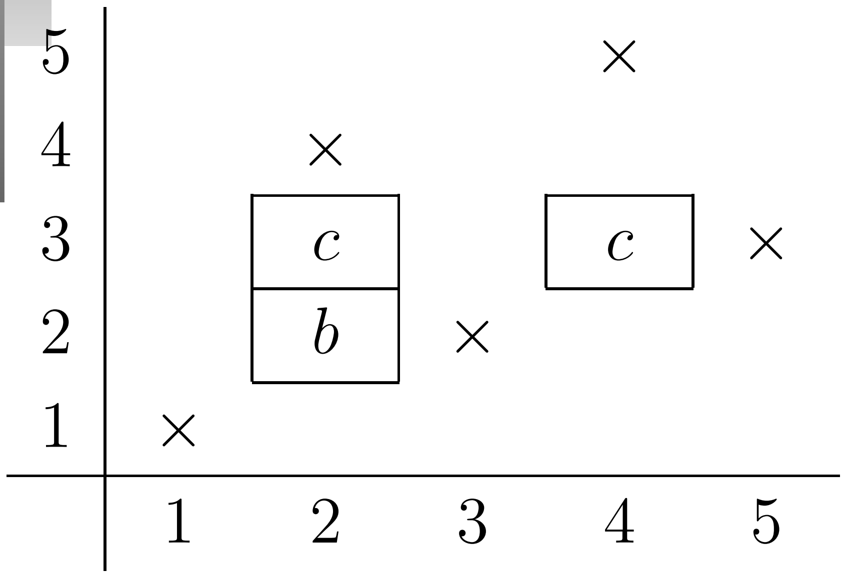

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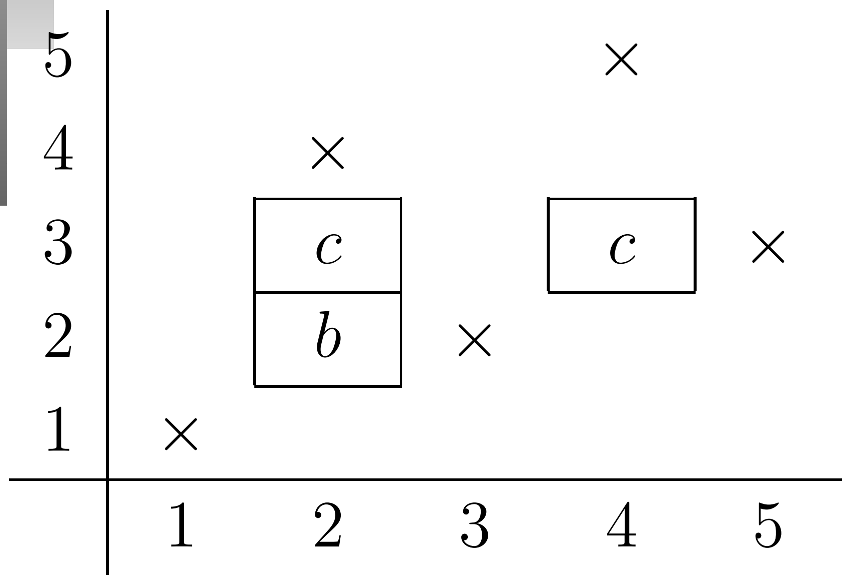

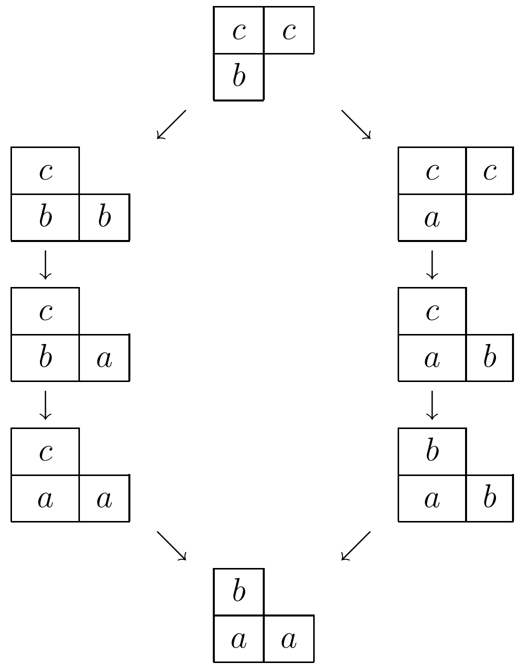

Multiplication

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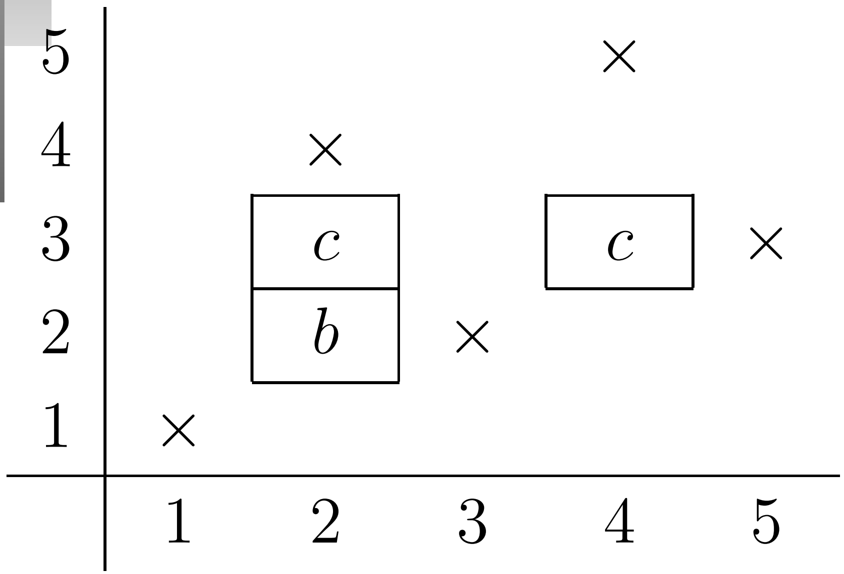

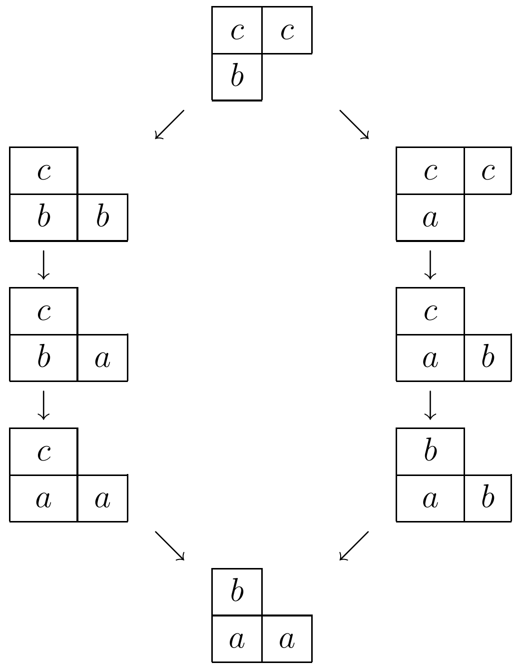

$$= S_{21}(A_3)$$

Schubert polynomial $X_{1246358\,10\,79}$

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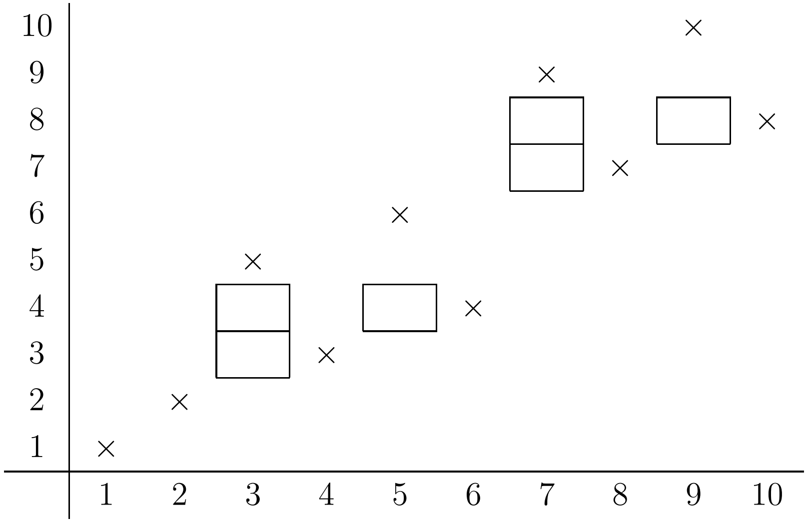

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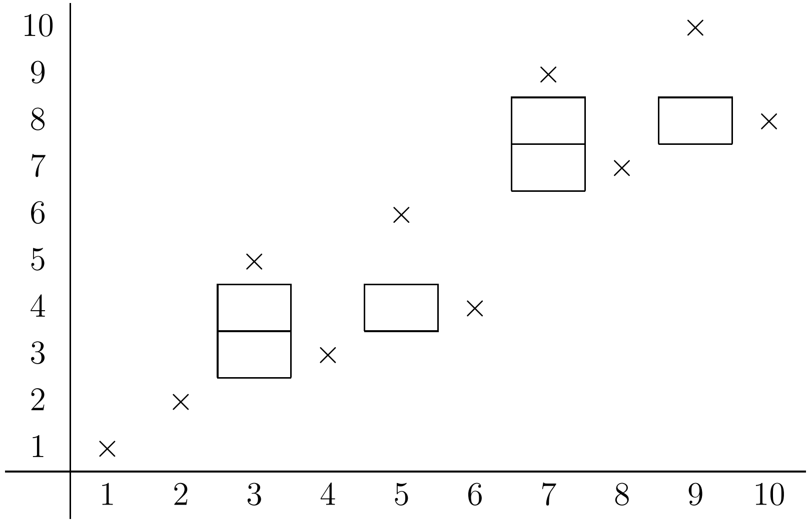
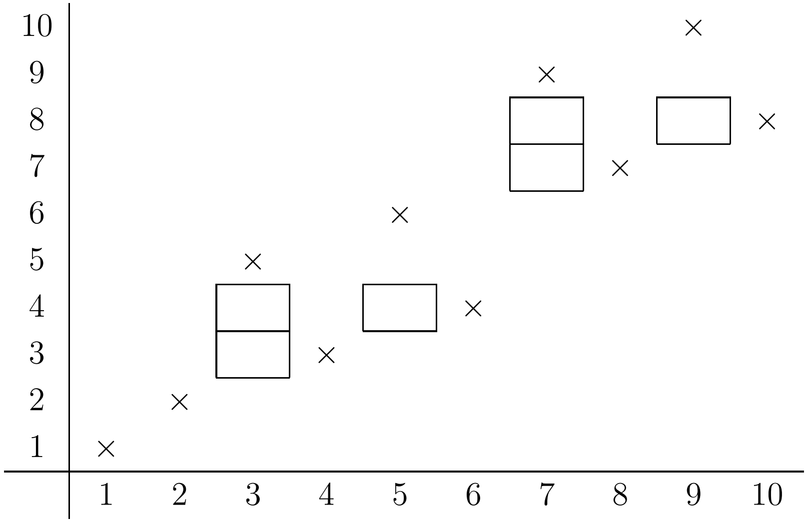


- $X_{1246358\,10\,79} = S_{21}(A_4) \times S_{21}(A_8)$

Multiplication

Schubert polynomial $X_{1246358\,10\,79}$



- $X_{1246358\,10\,79} = S_{21}(A_4) \times S_{21}(A_8)$
- $X_{1246358\,10\,79} \downarrow A_4 = S_{21}(A_4) \times S_{21}(A_4)$

Multiplication

Monk's rule $X_w \in \mathbb{N}[a_1, \ldots]$

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Multiplication

Monk’s rule $X_w \in \mathbb{N}[a_1, \ldots]$

$$a_i \times X_w = \sum_{w'} X_{w'} - \sum_{w''} X_{w''}$$

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Multiplication

Monk's rule $X_w \in \mathbb{N}[a_1, \ldots]$

$$a_i \times X_w = \sum_{w'} X_{w'} - \sum_{w''} X_{w''}$$

- $w' = w(i,j)$ with $j > i$ and $l(w') = l(w) + 1$

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Multiplication

Monk's rule $X_w \in \mathbb{N}[a_1, \ldots]$

$$a_i \times X_w = \sum_{w'} X_{w'} - \sum_{w''} X_{w''}$$

- $w' = w(i,j)$ with $j > i$ and $l(w') = l(w) + 1$
- $w'' = w(i,j)$ with $j < i$ and $l(w'') = l(w) + 1$

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Monk's rule $X_{1246358\,10\,79}$

$a_7 \times X_{1246358\,10\,79}$=

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$a_7 \times X_{1246358\,10\,79} = X_{124635\,10\,879} + X_{1246359\,10\,78}$

Multiplication

Monk's rule $X_{1246358\,10\,79}$

$a_7 \times X_{124635{\color{red}8}\,10\,79} = X_{124635\,10\,879} + X_{1246359\,10\,78}$

$- X_{1246385\,10\,79} - X_{1248356\,10\,79}$

Monk's rule $X_{1246358\,10\,79}$

$a_7 \times X_{1246358\,10\,79} = X_{124635\,10\,879} + X_{1246359\,10\,78}$

$- X_{1246385\,10\,79} - X_{1248356\,10\,79}$

$a_8 \times X_{12483569 7\,10} =$

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Monk's rule $X_{1246358\,10\,79}$

$a_7 \times X_{1246358\,10\,79} = X_{124635\,10\,879} + X_{1246359\,10\,78}$

$- X_{1246385\,10\,79} - X_{1248356\,10\,79}$

$a_8 \times X_{1248356\,9\,7\,10} = X_{1248356\,10\,79}$

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$a_7 \times X_{1246358\,10\,79} = X_{124635\,10\,879} + X_{1246359\,10\,78}$

$- X_{1246385\,10\,79} - X_{1248356\,10\,79}$

$a_8 \times X_{1248356\,97\,10} = X_{1248356\,10\,79}$

$- X_{124835967\,10} - X_{124935687\,10}$

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Monk's rule $X_{1246358\,10\,79}$

$a_7 \times X_{1246358\,10\,79} = X_{124635\,10\,879} + X_{1246359\,10\,78}$
$- X_{1246385\,10\,79} - X_{1248356\,10\,79}$

$a_8 \times X_{12483569 7\,10} = X_{1248356\,10\,79}$
$- X_{124835967\,10} - X_{124935687\,10}$

- i index of last decrease in w :
  $a_i \times X_w = X_{w'} - \sum X_{w''}$

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- i index of last decrease in w :
  $a_i \times X_w = X_{w'} - \sum X_{w''}$
- $X_{1248356\,10\,79} = a_8 \times X_{124835697} + X_{124835967} + X_{124935687}$

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Littlewood Richardson - Lascoux Schützenberger

$$1246358\ 10\ 79$$

$$0012001200$$

$$a_8 \times X_{124635897} + X_{124635987}$$

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Littlewood Richardson - Lascoux Schützenberger

$$\begin{array}{c} 1246358\ 10\ 79 \\ 0012001200 \\ \downarrow \\ 124635987 \\ 001200210 \end{array}$$

$$a_8 \times X_{124635978} + X_{124637958} + X_{124735968}$$

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Littlewood Richardson - Lascoux Schützenberger

1246358 10 79

0012001200

↓

124635987

001200210

↙ ↘

124637958   124735968

001201200   001300200

$$a_7 \times X_{124735869} + X_{124738569} + X_{124835769}$$

Littlewood Richardson - Lascoux Schützenberger

1246358 10 79
0012001200

↓

124635987
001200210

↙ ↘

124637958
001201200

124735968
001300200

↙ ↘

12473856
00130200

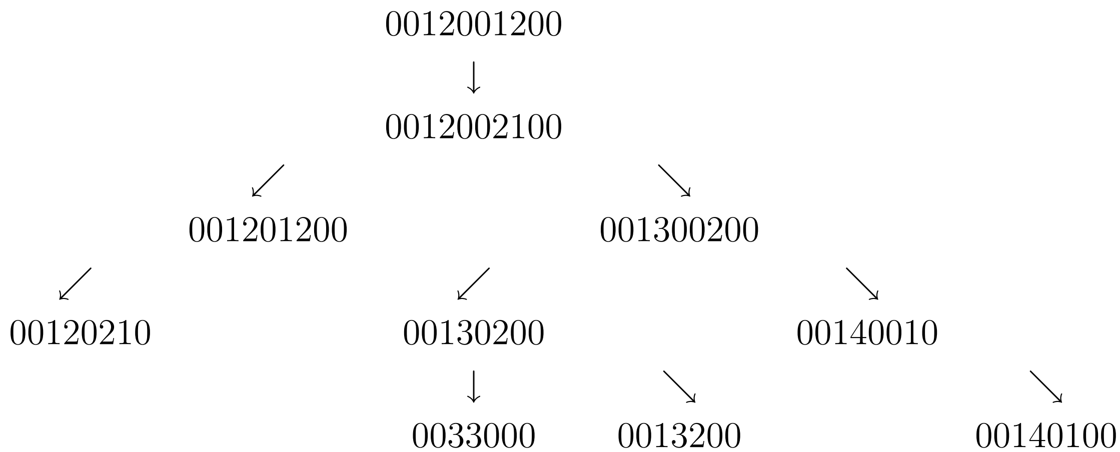
12483576
00140010

Multiplication

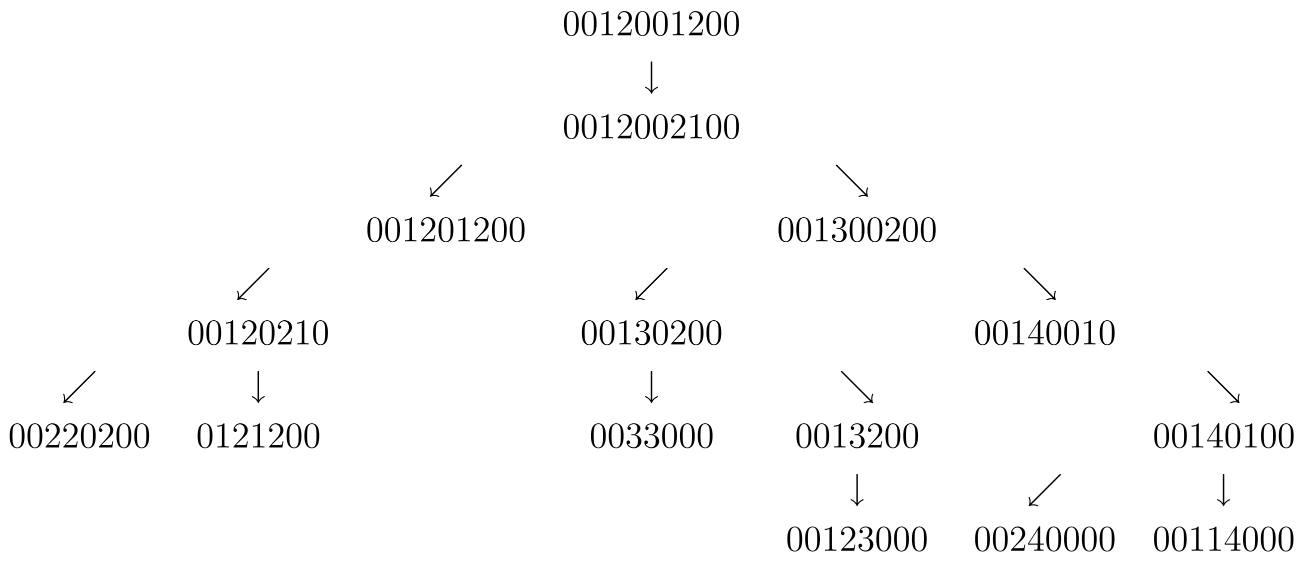
Lascoux Schützenberger

$$0012001200$$
$$\downarrow$$
$$0012002100$$
$$\swarrow \qquad\qquad\qquad \searrow$$
$$001201200 \qquad\qquad 001300200$$
$$\swarrow \qquad\qquad\qquad \searrow$$
$$00130200 \qquad\qquad 00140010$$

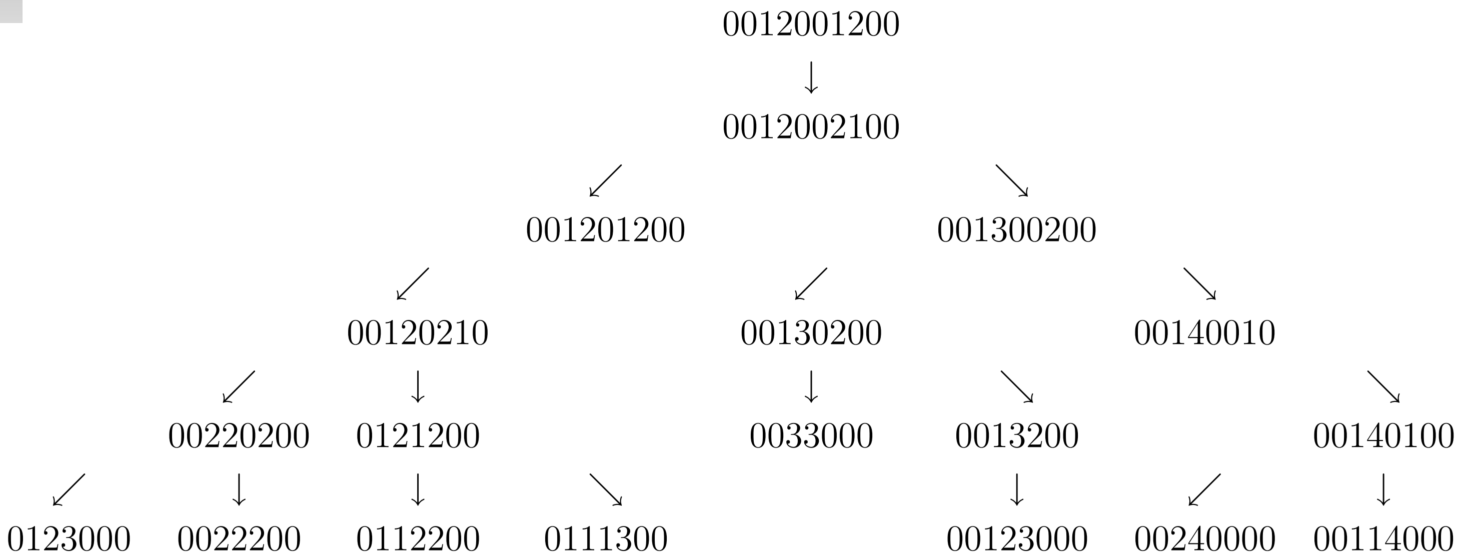
Lascoux Schützenberger



Lascoux Schützenberger



Lascoux Schützenberger



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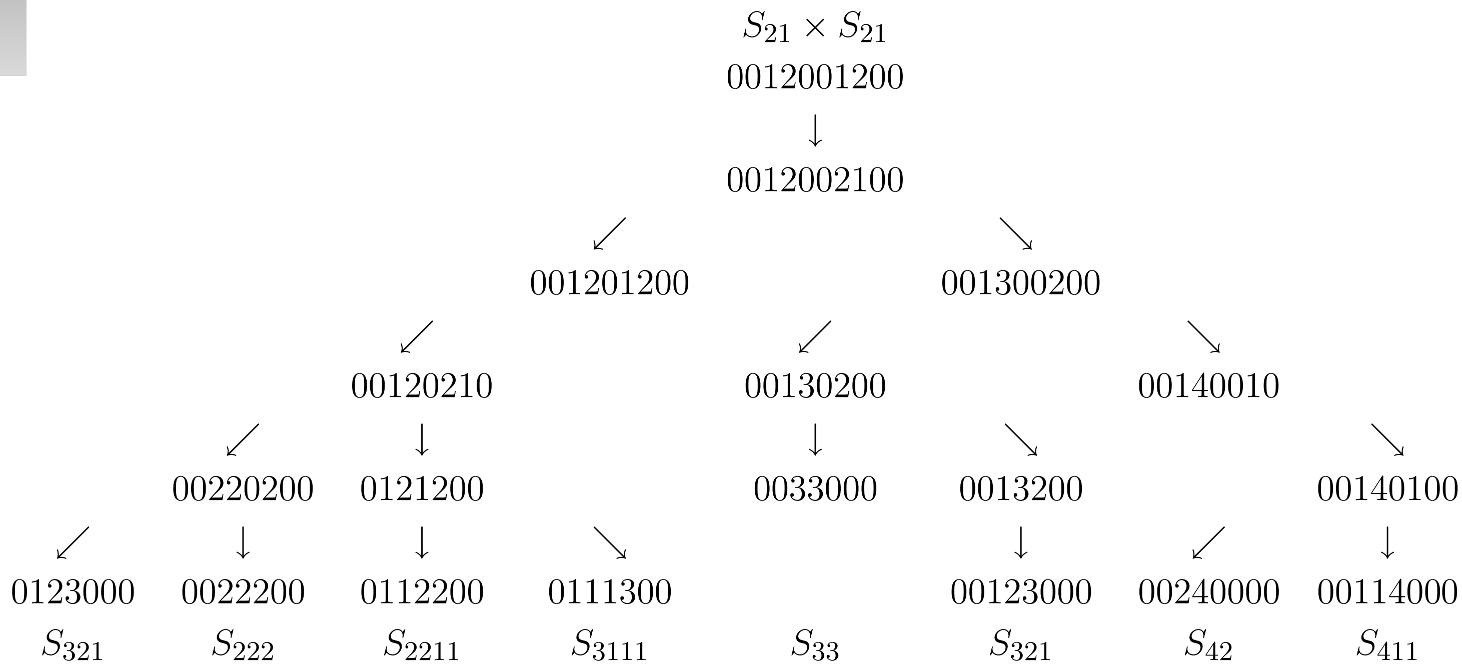
Lascoux Schützenberger



Multiplication

MAGMA

```
>Q := Rationals();
>S := SFASchur(Q);
>s21 := S.[2,1];
>s21*s21;
S.[2,2,1,1] + S.[2,2,2] + S.[3,1,1,1] + 2*S.[3,2,1]
+ S.[3,3] + S.[4,1,1] + S.[4,2]
```

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third operation on Λ

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- given f, g of degree m, n

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- addition $f + g$ of degree $\sim$max(m, n)

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third operation on Λ

- given f, g of degree m, n
- addition $f + g$ of degree $\sim$max(m, n)
- multiplication $f \times g$ of degree $n + m$
- plethysm $f[g]$ of degree $m \times n$

easiest case: e.g. $h_2[e_2]$

easiest case: e.g. $h_2[e_2]$

- $h_2 = \boxed{a}\boxed{a} + \boxed{a}\boxed{b} + \boxed{a}\boxed{c} + \ldots$
- $e_2 = \begin{array}{|c|}\hline b \\ \hline a \\ \hline \end{array} + \begin{array}{|c|}\hline c \\ \hline a \\ \hline \end{array} + \begin{array}{|c|}\hline d \\ \hline a \\ \hline \end{array} + \ldots$

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- $h_2 = \boxed{a}\boxed{a} + \boxed{a}\boxed{b} + \boxed{a}\boxed{c} + \ldots$
- $e_2 = \begin{array}{|c|}\hline b \\ \hline a \\ \hline\end{array} + \begin{array}{|c|}\hline c \\ \hline a \\ \hline\end{array} + \begin{array}{|c|}\hline d \\ \hline a \\ \hline\end{array} + \ldots$
- variables in h_2 = tableaux of e_2

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easiest case: e.g. $h_2[e_2]$

- $h_2 = \begin{array}{|c|c|}\hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline a & b \\ \hline\end{array} + \begin{array}{|c|c|}\hline a & c \\ \hline\end{array} + \ldots$
- $e_2 = \begin{array}{|c|}\hline b \\ \hline a \\ \hline\end{array} + \begin{array}{|c|}\hline c \\ \hline a \\ \hline\end{array} + \begin{array}{|c|}\hline d \\ \hline a \\ \hline\end{array} + \ldots$
- variables in h_2 = tableaux of e_2
- $h_2[e_2] = \begin{array}{|c|c|}\hline b & b \\ \hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline b & c \\ \hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array} + \ldots$

easiest case: e.g. $h_2[e_2]$

- $h_2[e_2] = \begin{array}{|c|c|}\hline b & b \\ \hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline b & c \\ \hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array} + \dots$

  was misleading

Plethysm

easiest case: e.g. $h_2[e_2]$

- $h_2[e_2] = \begin{array}{|c|c|}\hline b & b \\ \hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline b & c \\ \hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array} + \ldots$

  was misleading

- $h_2[e_2] = \begin{array}{|c|c|}\hline b & b \\ \hline a & a \\ \hline\end{array} \ldots \begin{array}{|c|c|}\hline c & d \\ \hline a & b \\ \hline\end{array} \ldots \begin{array}{|c|c|}\hline d & c \\ \hline a & b \\ \hline\end{array} \ldots$

easiest case: e.g. $h_2[e_2]$

- $h_2[e_2] = \begin{array}{|c|c|}\hline b & b \\ \hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline b & c \\ \hline a & a \\ \hline\end{array} + \begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array} + \dots$

  was misleading

- $h_2[e_2] = \begin{array}{|c|c|}\hline b & b \\ \hline a & a \\ \hline\end{array} \dots \begin{array}{|c|c|}\hline c & d \\ \hline a & b \\ \hline\end{array} \dots \begin{array}{|c|c|}\hline d & c \\ \hline a & b \\ \hline\end{array} \dots$

- $h_2[e_2] = S_2[S_{11}] = S_{22} + S_{1111}$

MAGMA

```
>Q := Rationals();
>S := SFASchur(Q);
>e2 := S.[1,1];
>h2:=S.[2];
>h2~e2;
S.[1,1,1,1] + S.[2,2]
```

algorithm for $S_n[S_m] = \sum \ldots S_K$

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- $S_n[S_m]$= 'plethystic' fillings of $n \times m$ rectangle

Plethysm

algorithm for $S_n[S_m] = \sum \ldots S_K$

- $S_n[S_m]$= 'plethystic' fillings of $n \times m$ rectangle
- $S_3[S_2] =$

aa	aa	aa	aa	aa	ab	aa	aa	ac	aa	aa	bb	aa	aa	bc	aa	aa	cc	... =

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algorithm for $S_n[S_m] = \sum \ldots S_K$

- $S_n[S_m]$= 'plethystic' fillings of $n \times m$ rectangle
- $S_3[S_2] =$

aa	aa	aa

aa	aa	ab

aa	aa	ac

aa	aa	bb

aa	aa	bc

aa	aa	cc

... =

a	a
a	a
a	a

a	b
a	a
a	a

a	c
a	a
a	a

b	b
a	a
a	a

b	c
a	a
a	a

c	c
a	a
a	a

...

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Plethysm

algorithm for $S_n[S_m] = \sum \ldots S_K$

- $S_n[S_m]$= 'plethystic' fillings of $n \times m$ rectangle
- $S_3[S_2] =$

aa	aa	aa

aa	aa	ab

aa	aa	ac

aa	aa	bb

aa	aa	bc

aa	aa	cc

... =

a	a
a	a
a	a

a	b
a	a
a	a

a	c
a	a
a	a

b	b
a	a
a	a

b	c
a	a
a	a

c	c
a	a
a	a

...

- $=: P_{222} = P_{2^3}.$

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Plethysm

for arbitrary partitions $I = k^{m_k}$,...,1^{m_1}

- $P_I = \prod P_{i^{m_i}}$

for arbitrary partitions $I = k^{m_k}, \ldots, 1^{m_1}$

- $P_I = \prod P_{i^{m_i}}$
- $P_{m^n}(a, b, c, \ldots) = \sum_{i=0}^{n \cdot m} a^i \sum_{|I|=nm-i, I \subseteq m^n} P_I(b, c, d, \ldots)$

Plethysm

for arbitrary partitions $I = k^{m_k}, \dots, 1^{m_1}$

- $P_I = \prod P_{i^{m_i}}$
- $P_{m^n}(a, b, c, \dots) = \sum_{i=0}^{n \cdot m} a^i \sum_{|I| = nm - i, I \subseteq m^n} P_I(b, c, d, \dots)$
-

a	a
a	a
a	a

a	
a	a
a	a

a	
a	
a	a

a	a
a	a

a	
a	a

a	
a	
a	

a	
a	

a	a

a	

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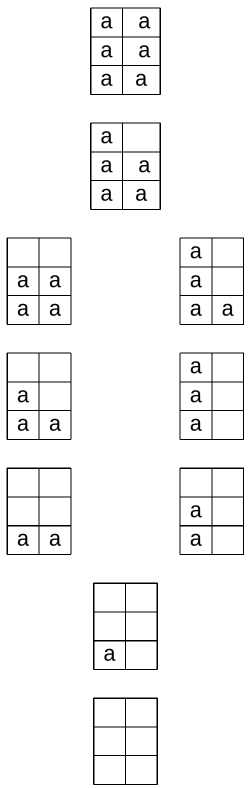
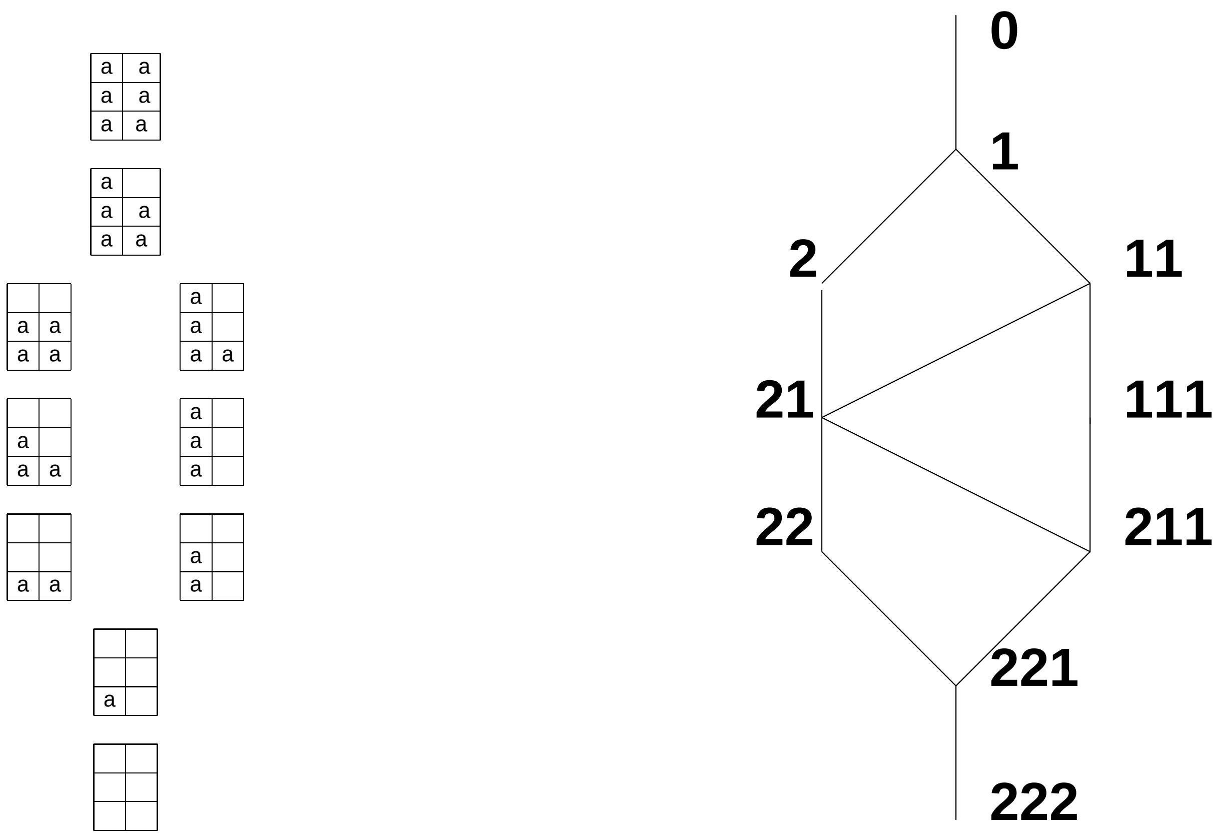
Plethysm



Plethysm



Plethysm

- first step:

a	a
a	a
a	a

$\in S_3[S_2] \Rightarrow S_6 \in S_3[S_2]$

Plethysm

- first step:

a	a
a	a
a	a

$\in S_3[S_2] \Rightarrow S_6 \in S_3[S_2]$

- second step:

a	P_1
a	a
a	a

$P_1 = S_1$

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- first step:

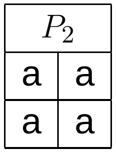
$$\begin{array}{|c|c|}\hline a & a \\ \hline a & a \\ \hline a & a \\ \hline \end{array} \in S_3[S_2] \Rightarrow S_6 \in S_3[S_2]$$

- second step:

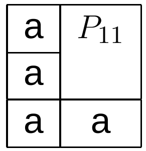
$$\begin{array}{|c|c|}\hline a & P_1 \\ \hline a & a \\ \hline a & a \\ \hline \end{array} \; P_1 = S_1$$

- S_6 without $\begin{array}{|c|c|c|c|c|}\hline a & a & a & a & a \\ \hline \end{array}$ = S_1

- third step



$P_2 = S_2$



$P_{11} = S_2$ together $2S_2$

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- third step

P_2	
a	a
a	a

$P_2 = S_2$

a	P_{11}
a	
a	a

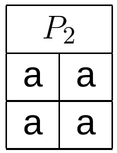
$P_{11} = S_2$ together $2S_2$

- S_6 without

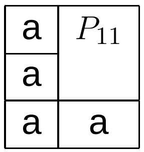
a	a	a	a

= S_2

- third step

  $P_2 = S_2$

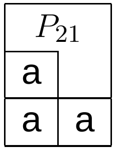
  

  $P_{11} = S_2$ together $2S_2$

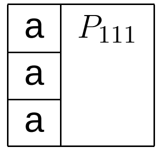
- S_6 without | a | a | a | a | = S_2

- add to the result: S_{42}

- fourth step



$P_{21} = S_{21} + S_3$

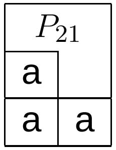


$P_{111} = S_3$ together

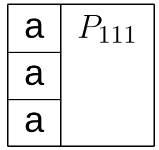
$2S_3 + S_{21}$

Plethysm

- fourth step



$P_{21} = S_{21} + S_3$



$P_{111} = S_3$ together $2S_3 + S_{21}$

- $S_6 + S_{42}$without | a | a | a | = $2S_3 + S_{21}$

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$S_3[S_2]$

level	result	current	needed	new
5	S_6	$S_{6/5} = S_1$	S_1	0
4	S_6	$S_{6/4} = S_2$	$2S_2$	S_{42}
3	$S_6 + S_{42}$	$S_{6/3} + S_{42/3} =$	$2S_3 + S_{21}$	0
		$2S_3 + S_{21}$		
2	$S_6 + S_{42}$	$S_{6/2} + S_{42/2} =$	$2S_4 + 2S_{22} + S_{31}$	S_{222}
		$2S_4 + S_{22} + S_{31}$		

$S_3[S_2] = S_6 + S_{42} + S_{222}$

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Transition Matrices

- 5 bases, fixed degree

Transition Matrices

- 5 bases, fixed degree
- matrix size = number of partitions

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Transition Matrices

- 5 bases, fixed degree
- matrix size = number of partitions
- combinatorial interpretation

Transition Matrices

- 5 bases, fixed degree
- matrix size = number of partitions
- combinatorial interpretation
- power sum $\rightarrow$ Schur : character table Sym_n

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MAGMA

```
>PowerSumToSchurMatrix(5);
[ 1 -1 0 1 0 -1 1]
[ 1 0 -1 0 1 0 -1]
[ 1 -1 1 0 -1 1 -1]
[ 1 1 -1 0 -1 1 1]
[ 1 0 1 -2 1 0 1]
[ 1 2 1 0 -1 -2 -1]
[ 1 4 5 6 5 4 1]
```

MAGMA

```
>S := SFASchur(Rationals());
>P:=SFAPower(Rationals());
>S!P.[1,1,1,1,1];
S.[1,1,1,1,1] + 4*S.[2,1,1,1] + 5*S.[2,2,1]
+ 6*S.[3,1,1] + 5*S.[3,2] + 4*S.[4,1] + S.[5]
```

Transition Matrices

different problems

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different problems

- computation of a complete matrix
  use conjugate partition

different problems

- computation of a complete matrix
  use conjugate partition

- computation of a single row

Transition Matrices

different problems

- computation of a complete matrix
  use conjugate partition

- computation of a single row

- computation of a single value
  Murnaghan Nakayama rule

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Thank you very much for your attention.